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THE ADMITTANCE OF APERTURE ANTENNAS RADIATING INTO LOSSY MEDIA

by

R.T. Compton, Jr.

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Department of ELECTRICAL ENGINEERING

THE OHIO STATE UNIVERSITY
RESEARCH FOUNDATION
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REPORT

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ABSTRACT

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This report is concerned with the radiation of aperture antennas into lossy media. Two antennas are considered: the infinite slot in a ground plane and the rectangular aperture in a ground plane. The object of the study is to determine for each antenna the admittance as a function of the electrical constants of the lossy medium. For each antenna configuration, two problems are treated. First, the antenna is assumed to radiate into a lossy half-space, and second, the antenna is assumed to radiate through a lossy slab into a free-space region. For each case the aperture admittance is calculated as a function of the complex propagation constant of the lossy medium and the aperture dimensions.

Author

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CHAPTER I
INTRODUCTION

This study is concerned with the radiation of aperture antennas into lossy media. Two types of antennas are considered: the infinite slot in a ground plane and the rectangular aperture in a ground plane. The object of the study is to determine for each antenna the admittance as a function of the electrical constants of the lossy medium.

This work is motivated in part by a need to measure the properties of a plasma by means of the admittance of an antenna in the plasma. The rectangular aperture in particular is a practical form of antenna for such experiments.

The properties of antennas in lossy media have been studied by many authors. The earliest work was done shortly before and during World War I, when submarine communication problems stimulated interest in submerged antennas. Geophysical prospecting also provided motivation for some of the work. By far the greatest portion of the work on lossy media, however, was done after World War II.

Extensive bibliographies on this subject have been given by Hansen,[1] Moore,[2] and Tai.[3]

The most notable contribution during the early period was of course Sommerfeld's treatment of the dipole over a lossy half-space. This work is given in Sommerfeld,[4] Stratton,[5] and Bremmer.[6] Other work during the early period indudes that of Carson,[7] Foster,[8]

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[1] R.C. Hansen, "Radiation and Reception with Buried and Submerged Antennas," Trans. of the IEEE, Vol. AP-11, No. 3, p. 207, May, 1963.

[2] R.K. Moore, "Effects of a Surrounding Conducting Medium on Antenna Analysis," Trans. of the IEEE, Vol. AP-11, No. 3, p. 216, May, 1963.

[3] C.T. Tai, "Antennas in Lossy Media," Proc. of the 1963 URSI General Assembly.

[4] A. Sommerfeld, Partial Differential Equations in Physics, Academic Press, Inc., New York, p. 236, 1949.

[5] J.A. Stratton, Electromagnetic Theory, McGraw-Hill Book Co., New York, p. 573, 1941.

[6] H. Bremmer, "Electric Fields and Waves," in Handbuch der physik, S. Flugge, Ed., Springer Verlag, Berlin, Germany, Vol. 16, Chapt. 4, p. 519, 1958.

[7] J.R. Carson, "Wave Propagation in Overhead Wires with Ground Return," Bell Sys. Tech. Jour., Vol. 5, pp. 539-554, October, 1926.

[8] R.M. Foster, "Mutual Impedance of Grounded Wires Lying on the Surface of the Earth," Bell Sys. Tech. Jour., Vol. 10, pp. 408-419, July, 1931.

Riordan and Sunde,[9] who were concerned with the effect of a lossy earth on overhead wires and wires lying on the ground. Batcher,[10] Bouthillon,[11] Taylor,[12,13] Willoughby and Lowell[14] considered the submarine communication problem.

The studies on lossy media appearing after World War II are much more numerous. Since our primary interest here is in the admittance properties of antennas, we will discuss only those papers dealing with the admittance. C.T. Tai in 1947 considered an infinitesimal dipole in an infinite lossy medium and found that the input

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[9] J. Riordan and E.D. Sunde, "Mutual Impedance of Grounded Wires for Horizontally Stratified Two-Layer Earth," Bell Sys. Tech. Jour., Vol. 12, pp. 162-177, April 1933.

[10] R.R. Batcher, "Loop Antennas for Submarines," Wireless Age, Vol. 7. p. 28, 1920.

[11] L. Bouthillon, "Contribution a l'etude des Radio communications Sousmarines," Rev. gen. 'elect., Vol. 7. pp. 696-700, May, 1920.

[12] Taylor, A.H., "Short Wave Reception and Transmission on Ground Wires (Subterranean and Submarine)," Proc. IRE, Vol. 7, pp. 337-362, August, 1919.

[13] A.H. Taylor, "Long Wave Reception and the Elimination of Strays on Ground Wires (Subterranean and Submarine)," Proc. IRE, Vol. 7, pp. 559-583, December, 1919.

[14] J.A. Willoughby and P.D. Lowell, "Development of Loop Aerials for Submarine Radio Communication," Phys. Rev., Vol. 14, pp. 193-194, August, 1919.

power is infinite.[15] Tai also found the input impedance of a biconical antenna in an insulating cavity.[16] Wait has discussed the impedance of a long wire over the earth,[17] and the mutual impedance of loops lying on the earth.[18] He has also determined the radiation resistance of a horizontal loop over the earth[19] and the radiation resistance of dipoles in the interface between two dielectrics.[20] King and Harrison[21] have discussed the impedance of a half-wave cylindrical antenna in a dissipative medium. An extensive treatment of the

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[15] C.T. Tai, "Radiation of a Hertzian Dipole Immersed in a Conducting Medium," Cruft Lab., Harvard University, Cambridge, Mass., Rep. No. 21, October, 1947.

[16] C.T. Tai, "On Radiation and Radiation Systems in the Presence of a Dissipative Medium," Cruft Lab., Harvard University, Cambridge, Mass., Rept. No. 77, October, 1949.

[17] J.R. Wait, "On the Impedance of a Long Wire Suspended over the Ground," Proc. IRE, Vol. 49, p. 1576, October, 1961.

[18] J.R. Wait, "Mutual Coupling of Loops Lying on the Ground," Geophys., Vol. 19, pp. 290-296, April, 1954.

[19] J.R. Wait, Radiation Resistance of a Small Circular Loop in the Presence of a Conducting Ground," J. Appl. Phys., Vol. 24, pp. 646-649, May, 1953.

[20] J.R. Wait, "Radiation Resistance of Dipoles in an Interface between Two Dielectrics," Can. J. Phys., Vol. 34, pp. 24-26, January, 1956.

[21] R.W.P. King and C.W. Harrison, "Half-Wave Cylindrical Antenna in a Dissipative Medium: Current and Impedance," J. Res. NBS, Vol. 64D, pp. 365-380, July-August, 1960.

cylindrical antenna in a lossy medium has also been given by King.[22] Kraichman has discussed the impedance of a circular loop in an infinite lossy medium.[23]

Some recent work has been done by Guy and Hasserjian[24] who have discussed the impedance of dipoles for an array immersed in a lossy half-space. Ghose[25] has examined the mutual coupling between subsurface antenna array elements. Bhattacharyya[26] has investigated the resistances of horizontal electric and vertical magnetic dipoles over a lossy earth. Chen and King[27] have treated a

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[22] R.W.P. King, "Dipoles in Dissipative Media," Proc. Symp. on Electromagnetic Waves, Math. Research Center, Univ. of Wisconsin, Madison, Wis., April, 1961, Univ. of Wisconsin Press, pp. 199-241, 1962.

[23] M.B. Kraichman, "Impedance of a Circular Loop in an Infinite Conducting Medium," J. Res. NBS, Vol. 66D, pp. 499-503, July-August, 1962.

[24] A.W. Guy and G. Hasserjian, "Impedance Properties of Large Subsurface Antenna Arrays," Trans. of the IEEE, Vol. AP-11, No. 3, pp. 232-240, May, 1963.

[25] R.N. Ghose, "Mutual Couplings Among Subsurface Antenna Array Elements," Trans. of the IEEE, Vol. AP-11, No. 3, pp. 257-261, May, 1963.

[26] B.K. Bhattacharyya, "Input Resistances of Horizontal Electric and Vertical Magnetic Dipoles Over a Homogeneous Ground," Trans. of the IEEE, Vol. AP-11, No. 3, pp. 261-266, May, 1963.

[27] C.L. Chen and R.W.P. King, "The Small Bare Loop Antenna Immersed in a Dissipative Medium," Trans. of the IEEE, Vol. AP-11, No. 3, pp. 266-269, May, 1963.

bare loop antenna in a dissipative medium. Fenwick and Weeks[28] have discussed impedance of buried insulated wires.

Finally, the work of Galejs should be noted. Galejs has considered the admittances of slot antennas radiating through a plasma layer. He has used the clever technique of considering the infinite half-space to be a waveguide of very large, but finite, dimensions, allowing the fields to be represented by a discrete sum of modes.[29,30,31]

As mentioned above, in this study two types of apertures are considered, the infinite slot in a ground plane and the rectangular aperture in a ground plane.

In Chapter II, an expression for the admittance of an arbitrary aperture is derived and shown to be stationary. In Chapter III the

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[28] R.C. Fenwick and W.L. Weeks, "Submerged Antenna Characteristics," Trans. of the IEEE, Vol. AP-11, No. 3, pp. 296-305, May, 1963.

[29] J. Galejs, "Admittance of Annular Slot Antennas Radiating into a Plasma Layer," Project No. 125, Applied Research Laboratory, Sylvania Electronic Systems, 30 July, 1963.

[30] J. Galejs, "Admittance of a Waveguide Radiation into Stratified Plasma," Proj. No. 125, Appl. Res. Lab., Sylvania Elec. Sys., June, 1963.

[31] J. Galejs, "Slot Antenna Impedance for Plasma Layers," Appl. Res. Labs., Sylvania Elec. Sys., 23, July, 1963.

admittance of an infinite slot is found for two cases: radiating into an infinite lossy medium and radiating into a lossy slab. Chapter IV treats the rectangular aperture, also for the same two cases.

CHAPTER II
THE ADMITTANCE OF AN APERTURE

Consider an aperture which opens through an infinitely conducting ground plane into a half-space region, as shown in Fig. 1.

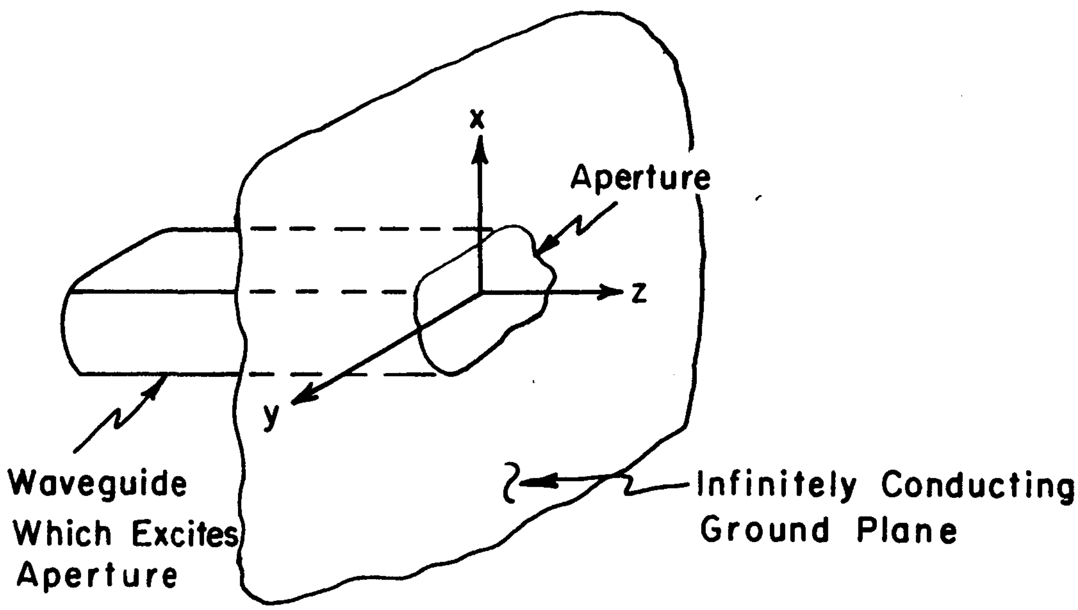

Fig. 1--Aperture in a ground plane

The half-space $z > 0$ consists of an isotropic dielectric with arbitrary permittivity ϵ_1 and conductivity σ_1 and with free-space permeability. The fields in the aperture are excited by a waveguide whose cross-section is for the moment arbitrary except that it does not vary in the z direction. The waveguide is driven by a source which produces only a single mode incident on the aperture. The purpose of this section will be to derive a stationary formula for the admittance which effectively terminates this mode as a result of the aperture radiating into the half-space.

With a single mode incident on the aperture, the electric and magnetic fields in the aperture may be written

$$\bar{E}(x, y, o) = V_o \bar{e}_o(x, y) + \sum_{n=0}^{\infty} V_{rn} \bar{e}_n(x, y) \tag{1}$$

$$\bar{H}(x, y, o) = Y_o V_o \bar{h}_o(x, y) - \sum_{n=0}^{\infty} Y_n V_{rn} \bar{h}_n(x, y) \tag{2}$$

where $\bar{e}_n(x, y)$ and $\bar{h}_n(x, y)$ are the transverse vector mode functions[32] appropriate to the particular waveguide cross-section and satisfying the relations

$$\bar{e}_n(x, y) = \bar{h}_n(x, y) \times \hat{z} \tag{3}$$

$$\iint_{\text{aperture}} |\bar{e}_n(x, y)|^2 \, dx \, dy = \iint_{\text{aperture}} |\bar{h}_n(x, y)|^2 \, dx \, dy = 1 \; . \tag{4}$$

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[32] R.F. Harrington, Time Harmonic Electromagnetic Fields, McGraw-Hill Book, Co., New York, pp. 381ff, 1961.

In (1) and (2), V_o is the amplitude of the incident mode and V_{rn} is the complex amplitude of the n-th reflected mode. Y_n is the characteristic admittance of the waveguide for the n-th mode. The terminating admittance of the incident mode is given by

$$Y = Y_o \frac{V_o - V_{ro}}{V_o + V_{ro}} \quad . \tag{5}$$

Equation (5) may be put in a stationary form by the following manipulations. From Eq. (1) and the orthonormal properties of the vector mode functions

$$V_o + V_{ro} = \iint_{\text{aperture}} \bar{E}(x, y, o) \cdot \bar{e}_o(x, y) dx\, dy \tag{6}$$

$$V_{rn} = \iint_{\text{aperture}} \bar{E}(x, y, o) \cdot \bar{e}_n(x, y) dx\, dy \quad . \tag{7}$$

Hence Eq. (2) may be written in the form:

$$\bar{H}(x, y, o) + \sum_{n=1}^{\infty} Y_n \bar{h}_n(x, y) \iint_{\text{aperture}} \bar{E}(\eta, \xi, o) \cdot \bar{e}_n(\eta, \xi) d\eta\, d\xi = Y_o(V_o - V_{ro})\bar{h}_o(x, y), \tag{8}$$

or

$$\bar{\Gamma}(x, y) = Y_o(V_o - V_{ro})\bar{h}_o(x, y) \tag{9}$$

where $\bar{\Gamma}(x, y)$ is the quantity

$$(10) \qquad \overline{\Gamma}(x, y) = \overline{H}(x, y, o) + \sum_{n=1}^{\infty} Y_n \overline{h}_n(x, y) \iint_{\text{aperture}} \overline{E}(\eta, \xi, o) \cdot \overline{e}_n(\eta, \xi) d\eta\, d\xi \,.$$

Forming the vector product of $\overline{E}(x, y, o)$ and Eq. (9) and integrating the z-component of the result over the aperture yields the formula

$$(11) \qquad \iint_{\text{aperture}} \overline{E}(x, y, o) \times \overline{\Gamma}(x, y) \cdot \hat{z}\, dx\, dy = Y_o(V_o - V_{ro})(V_o + V_{ro}) \,.$$

Hence, combining Eqs. (5), (6), and (11), the admittance is found to be formally given by:[33]

$$(12) \qquad Y = \frac{\displaystyle\iint_{\text{aperture}} \overline{E}(x,y,o) \times \overline{\Gamma}(x, y) \cdot \hat{z}\, dx\, dy}{\left[\displaystyle\iint_{\text{aperture}} \overline{E}(x, y, o) \cdot \overline{e}_o(x, y) dx\, dy\right]^2} \,.$$

If the actual aperture fields are known, they may be substituted into Eq. (12) and the admittance may be found. As usual, this is not much help, because the fields are not known. The usefulness of Eq. (12) stems from the fact that this expression is stationary with respect to variations of the electric field about its exact value,

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[33] This expression is essentially the same as that given by Lewin, Advanced Theory of Waveguides, Illife and Sons, Ltd., London, pp. 121-125, 1951.

as will be shown below. Hence, an approximation for $\overline{E}$ may be used to obtain a good estimate for Y.[34]

To show that Eq. (12) is stationary, let

(13) $$\overline{E} = \overline{E}_o + \delta\overline{E}$$

where $\overline{E}_o$ is the exact aperture field and $\delta\overline{E}$ is the variation. Also, to simplify the algebra, let

(14) $$D_o = \iint_{\text{aperture}} \overline{E}_o(x, y, o) \cdot \overline{e}_o(x, y)\,dx\,dy$$

and

(15) $$\delta D = \iint_{\text{aperture}} \delta\overline{E} \cdot \overline{e}_o\,dx\,dy\ .$$

Then the first variation in Y is given by

(16) $$\delta Y = \frac{1}{D_o^4}\left[D_o^2 \iint_{\text{aperture}} \delta\overline{E} \times \overline{\Gamma}_o \cdot \hat{z}\,dx\,dy + D_o^2 \iint_{\text{aperture}} \overline{E}_o \times \delta\overline{\Gamma} \cdot \hat{z}\,dx\,dy \right.$$
$$\left. -2D_o\delta D \iint_{\text{aperture}} \overline{E}_o \times \overline{\Gamma}_o \cdot \hat{z}\,dx\,dy\right]$$

where the notation

(17) $$\overline{\Gamma} = \overline{\Gamma}_o + \delta\overline{\Gamma}$$

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[34] C. L. Dolph, "A Saddle Point Characterization of the Schwinger Stationary Points in Exterior Scattering Problems," J. Sco. Indust. Appl. Math., Vol. 5, No. 3, pp. 89-104, 1957.

has also been used. Here $\overline{\Gamma}_o$ is the value obtained by using the exact fields in Eq. (10) and $\delta\overline{\Gamma}$ is the variation.

To show that δY is zero, the first step is to show that

$$\iint_{\text{aperture}} \delta\overline{E} \times \overline{\Gamma}_o \cdot \hat{z}\, dx\, dy = \iint_{\text{aperture}} \overline{E}_o \times \delta\overline{\Gamma} \cdot \hat{z}\, dx\, dy \tag{18}$$

in Eq. (16). From Eq. (10), it is easily found that

$$\iint_{\text{aperture}} [\delta\overline{E} \times \overline{\Gamma}_o - \overline{E}_o \times \delta\overline{\Gamma}] \cdot \hat{z}\, dx\, dy = \iint_{\text{aperture}} [\delta\overline{E} \times \overline{H}_o - \overline{E}_o \times \delta\overline{H}] \cdot \hat{z}\, dx\, dy. \tag{19}$$

Now consider the fields in the region $z > 0$, the half-space. In this region both the trial fields, $\overline{E}$, $\overline{H}$ and the exact fields $\overline{E}_o$, $\overline{H}_o$ satisfy the Maxwell equations

$$\nabla \times \overline{E}_o = -j\omega\mu_o\overline{H}_o \tag{20}$$

$$\nabla \times \overline{H}_o = (j\omega\epsilon_1 + \sigma_1)\overline{E}_o \tag{21}$$

$$\nabla \times \overline{E} = -j\omega\mu_o\overline{H} \tag{22}$$

$$\nabla \times \overline{H} = (j\omega\epsilon_1 + \sigma_1)\overline{E} \tag{23}$$

and the radiation condition at large distance from the aperture. Hence from superposition the variations $\delta\overline{E}$, $\delta\overline{H}$ also satisfy Maxwell's equations and the radiation condition at large distance. Also, the tangential components of both $\overline{E}_o$ and $\delta\overline{E}$ are assumed to be zero on the $z = 0$ plane outside the aperture. Therefore

$$(24) \qquad \iint_{\text{aperture}} [\delta\overline{E} \times \overline{H}_o - \overline{E}_o \times \delta\overline{H}] \cdot \hat{z}\, dx\, dy = - \oiint_{\Sigma_1 + \Sigma_2} [\delta\overline{E} \times \overline{H}_o - \overline{E}_o \times \delta\overline{H}] \cdot \hat{n}\, ds$$

where Σ_1 is the $z = 0$ plane, Σ_2 is a spherical surface at infinite radius, and $\hat{n}$ is a unit vector normal to Σ_1 or Σ_2 and directed out of the enclosed volume of $\Sigma_1 + \Sigma_2$. But from the divergence theorem,

$$(25) \qquad \iint_{\Sigma_1 + \Sigma_2} [\delta\overline{E} \times \overline{H}_o - \overline{E}_o \times \delta\overline{H}] \cdot \hat{n}\, ds = \iiint_V \nabla \cdot [\delta\overline{E} \times \overline{H}_o - \overline{E}_o \times \delta\overline{H}] dv$$

$$= \iiint_V [\overline{H}_o \cdot \nabla \times \delta\overline{E} - \delta\overline{E} \cdot \nabla \times \overline{H}_o - \delta\overline{H} \cdot \nabla \times \overline{E}_o + \overline{E}_o \cdot \nabla \times \delta\overline{H}] dv$$

where "V" is the volume of the half-space $z > 0$. Using Maxwell's equations in (25) gives finally:

$$(26) \qquad \iint_{\Sigma_1 + \Sigma_2} [\delta\overline{E} \times \overline{H}_o - \overline{E}_o \times \delta\overline{H}] \cdot \hat{n}\, ds = \iiint_V [\overline{H}_o \cdot (-j\omega\mu_o \delta\overline{H}) - \delta\overline{E}$$

$$\cdot (j\omega\epsilon_1 + \sigma_1)\overline{E}_o - \delta\overline{H} \cdot (-j\omega\mu_o \overline{H}_o) + \overline{E}_o \cdot (j\omega\epsilon_1 + \sigma_1)\delta\overline{E}] dv = 0.$$

This establishes Eq. (18).[35] Using Eq. (18) in Eq. (16), δY may be written

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[35] In later sections two types of media will be considered for the region $z > 0$: an infinite lossy medium and a lossy slab of finite thickness. For the case of the slab, two volume integrals should be considered, one in the slab and one in the region outside the slab. The surface integrals over the slab interface are then equal because of the continuity of the electromagnetic field.

$$(27)\quad \delta Y = \frac{1}{D_o^4}\left[2D_o^2 \iint\limits_{\text{aperture}} \delta\bar{E} \times \bar{\Gamma}_o \cdot \hat{z}\, dx\, dy - 2D_o\,\delta D \iint\limits_{\text{aperture}} \bar{E}_o \times \bar{\Gamma}_o \cdot \hat{z}\, dx\, dy \right] .$$

Next, from Eqs. (16) and (14),

$$(28)\qquad V_{ro} = D_o - V_o$$

and hence Eq. (9) may be written

$$(29)\qquad \bar{\Gamma}(x, y) = Y_o(2V_o - D_o)\,\bar{h}_o(x, y) \quad .$$

Therefore

$$(30)\qquad \iint\limits_{\text{aperture}} \delta\bar{E} \times \bar{\Gamma}_o \cdot \hat{z}\, dx\, dy = Y_o(2V_o - D_o) \iint\limits_{\text{aperture}} \delta\bar{E} \times \bar{h}_o(x, y) \cdot \hat{z}\, dx\, dy \ .$$

Multiplying Eq. (30) by $2D_o^2$ and using Eq. (3) gives

$$(31)\qquad 2D_o^2 \iint\limits_{\text{aperture}} \delta\bar{E} \times \bar{\Gamma}_o \cdot \hat{z}\, dx\, dy = 2D_o^2\, Y_o(2V_o - D_o) \iint\limits_{\text{aperture}} \delta\bar{E} \cdot \bar{e}_o(x, y)\, dx\, dy$$

$$= 2D_o^2\, Y_o(2V_o - D_o)\,\delta D \ .$$

Also from Eq. (29)

$$(32)\qquad \iint\limits_{\text{aperture}} \bar{E}_o \times \bar{\Gamma}_o \cdot \hat{z}\, dx\, dy = Y_o(2V_o - D_o) \iint\limits_{\text{aperture}} E_o \times \bar{h}_o(x, y) \cdot \hat{z}\, dx\, dy$$

$$= Y_o(2V_o - D_o)\, D_o \ .$$

(Multiplying (32) by $2D_o\,\delta D$ yields

(33) $$2D_o \delta D \iint_{\text{aperture}} \overline{E}_o \times \overline{\Gamma}_o \cdot \hat{z}\, dx\, dy = 2D_o^2\, Y_o(2V_o - D_o)\,\delta D \, .$$

Equations (31) and (33), when inserted in (27), show that

(34) $$\delta Y = 0$$

which establishes the stationary nature of (12). In the following sections, Eq. (12) will be used to compute the admittance for several specific apertures.

CHAPTER III
THE INFINITE SLOT

In this chapter, the infinite slot antenna is treated. In Part A, the antenna is assumed to radiate into an infinite lossy medium. In Part B it is assumed to radiate through a lossy slab of finite thickness into free-space.

A. The Infinite Lossy Medium

Consider an infinite slot which opens through a ground sheet into a lossy half-space, as shown in Fig. 2. The slot extends infinitely far in the y-direction and has width "a" in the x-direction. Outside the slot, the xy-plane is infinitely conducting.

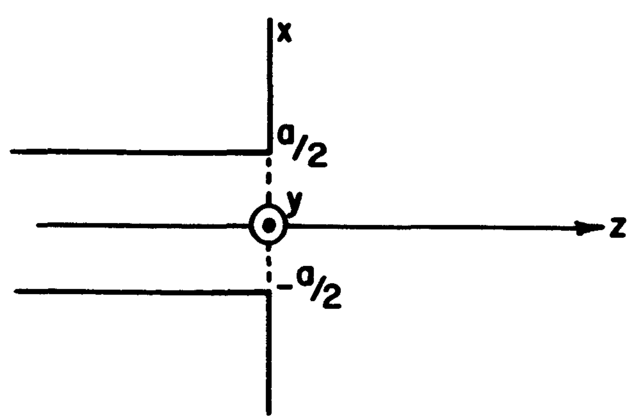

Fig. 2--Infinite slot in ground plane

MKS units and the time convention $e^{+j\omega t}$ will be used throughout this work.

The entire half-space $z > 0$ is assumed to be homogeneous and isotropic, and is characterized by a complex propagation constant

$$(35) \qquad k_1 = \left[\omega^2 \mu_o \epsilon_1 \left(1 - j\,\frac{\sigma_1}{\omega\epsilon_1}\right)\right]^{\frac{1}{2}}$$

where

k_1 = complex propagation constant

ω = radian frequency

μ_o = permeability of free-space

ϵ_1 = permittivity of $z > 0$ region

σ_1 = conductivity of $z > 0$ region.

The slot is excited by a parallel plate transmission line from behind the $z = 0$ plane. The electric field in the slot will be assumed to have the form of the fundamental mode for this structure,

$$(36) \qquad \overline{E}(x, y, o) = \overline{e}_o(x, y) = \begin{cases} \frac{1}{\sqrt{a}}\,\hat{x} : & |x| \leq a/2 \\ 0 \quad : & |x| > a/2 \end{cases} .$$

Equation (12) may now be applied to compute the admittance. For this geometry, all field quantities are independent of y and the appropriate form of Eq. (12) is

$$(37) \qquad Y = \frac{\displaystyle\int_{x=-a/2}^{a/2} \overline{E}(x, y, o) \times \overline{\Gamma}(x, y) \cdot \hat{z}\, dx}{\left[\displaystyle\int_{x=-a/2}^{a/2} \overline{E}(x, y, o) \cdot \overline{e}_o(x, y) dx\right]^2} \,.$$

Because of the orthogonality of the vector mode functions,

$$(38) \qquad \int_{-a/2}^{a/2} \overline{e}_o(x, y) \times \overline{h}_n(x, y) \cdot \hat{z}\, dx = 0 \quad (n \neq 0)$$

it is found that the integral in the numerator of (37) reduces to

$$(39) \qquad \int_{-a/2}^{a/2} \overline{E}(x, y, o) \times \overline{\Gamma}(x, y, o) \cdot \hat{z}\, dx = \int_{x=-a/2}^{a/2} \overline{E}(x, y, o) \times \overline{H}(x, y, o) \cdot \hat{z}\, dx \,.$$

Hence to evaluate Y the problem is to find the magnetic field $\overline{H}(x, y, o)$ resulting from $\overline{E}(x, y, o)$.

It is easily seen that the electromagnetic fields in this problem are everywhere TE to the y-axis. Hence the fields may be derived from a vector potential of the form

$$(40) \qquad \overline{F} = \hat{y}\, \psi$$

where ψ satisfies

$$(41) \qquad \nabla^2 \psi + k_1^2\, \psi = 0$$

with appropriate boundary conditions. The electric and magnetic fields are related to $\overline{F}$ through

(42) $$\overline{E} = -\nabla \times \overline{F}$$

(43) $$\overline{H} = \frac{1}{j\omega\mu_o}\left[k_1^2\overline{F} + \nabla(\nabla \cdot \overline{F})\right] .$$

Since the fields have no y-dependence, the $\overline{E}_x$ and $\overline{H}_y$ components are given by

(44) $$E_x = +\frac{\partial\psi}{\partial z}$$

(45) $$H_y = -(j\omega\epsilon_1 + \sigma_1)\psi .$$

This geometry may be regarded as a transmission line in the z-direction with infinite cross-section in the x-direction. Hence the solution for ψ may be constructed as the sum of a continuous spectrum of eigenvalues,

(46) $$\psi(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k_x)e^{-jk_z z}\, e^{-jk_x x}\, dk_x$$

where

(47) $$k_z = \sqrt{k_1^2 - k_x^2}$$

and the root is chosen so that

(48) $$\mathrm{Re}(k_z) \geq 0$$

(49) $$\mathrm{Im}(k_z) \leq 0$$

corresponding to propagation in the +z direction.

E_x is then given by

(50) $$E_x(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -jk_z\, f(k_x) e^{-jk_z z}\, e^{-jk_x x}\, dk_x$$

or in the z = 0 plane

(51) $$E_x(x, y, o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -jk_z f(k_x)\, e^{-jk_x x}\, dk_x \; .$$

The function $f(k_x)$ is then found by taking the inverse transform:

(52) $$-jk_z\, f(k_x) = \int_{-\infty}^{\infty} E_x(x, y, o)\, e^{+jk_x x}\, dk_x \; .$$

Using (36), this gives

(53) $$-jk_z\, f(k_x) = \int_{-a/2}^{a/2} \frac{1}{\sqrt{a}}\, e^{+jk_x x}\, dk_x = \frac{2}{\sqrt{a}\; k_x} \sin\left(\frac{k_x a}{2}\right)$$

or

(54) $$f(k_x) = \frac{2j}{\sqrt{a}\; k_x k_z} \sin\left(\frac{k_x a}{2}\right) \; .$$

Thus the solution for ψ is

$$(55) \qquad \psi(x, y, z) = \frac{1}{2\pi\sqrt{a}} \int_{-\infty}^{\infty} \frac{2j}{k_x k_z} \sin\left(\frac{k_x a}{2}\right) e^{-jk_z z} e^{-jk_x x} dk_x$$

and finally E_x and H_y are given by (44) and (45) as

$$(56) \qquad E_x(x, y, z) = \frac{1}{a\pi\sqrt{a}} \int_{-\infty}^{\infty} \sin \frac{2}{k_x} \left(\frac{k_x a}{2}\right) e^{-jk_z z} e^{-jk_x x} dk_x$$

$$(57) \qquad H_y(x, y, z) = \frac{-(j\omega\epsilon_1 + \sigma_1)}{2\pi\sqrt{a}} \int_{-\infty}^{\infty} \frac{2j}{k_x k_z} \sin\left(\frac{k_x a}{2}\right) e^{-jk_z z} e^{-jk_x x} dk_x.$$

For $z = 0$ these give

$$(58) \qquad E_x(x, y, o) = \frac{1}{2\pi\sqrt{a}} \int_{-\infty}^{\infty} \frac{2}{k_x} \sin\left(\frac{k_x a}{2}\right) e^{-jk_x x} dk_x$$

$$(59) \qquad H_y(x, y, o) = \frac{1}{2\pi\sqrt{a}} \int_{-\infty}^{\infty} \frac{2(\omega\epsilon_1 - j\sigma_1)}{k_x k_z} \sin \frac{k_x a}{2} e^{-jk_x x} dk_x.$$

With no y-component of the electric field, Eq. (39) becomes

$$(60) \qquad \int_{x=-a/2}^{a/2} \overline{E}(x, y, o) \times \overline{H}(x, y, o) \cdot \hat{z}\, dx = \int_{x=-a/2}^{a/2} E_x(x, y, o) H_y(x, y, o)\, dx$$

Since $E_x(x, y, o)$ is zero for $|x| > a/2$, the limits of integration in (60) can be extended to infinity. Then substituting (58) and (59) in

(60) and making use of Parseval's Theorem[36] gives

(61) $$\int_{x=-a/2}^{a/2} \overline{E}(x,y,o) \times \overline{H}(x,y,o) \cdot \hat{z}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\omega\epsilon_1 - j\sigma_1)}{k_x^2 k_z a} \sin^2\left(\frac{k_x a}{2}\right) dk_x \,.$$

Now the integrand may be rearranged and Parseval's Theorem may be used again. Let

(62) $$f_1(k_x) = \frac{4(\omega\epsilon_1 - j\sigma_1)}{k_z}$$

(63) $$f_2(k_x) = \frac{1}{ak_x^2} \sin^2\left(\frac{k_x a}{2}\right)$$

The transform of $f_1(k_x)$ is given by:

(64) $$F_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(k_x) e^{-jk_x x}\, dk_x$$

---

[36] For the transform pairs

$$F_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(k_x) e^{-jk_x x}\, dk_x$$

$$F_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(k_x) e^{-jk_x x}\, dk_x$$

Parseval's Theorem is:

$$\int_{-\infty}^{\infty} F_1(x) F_2^*(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(k_x) f_2^*(k_x)\, dk_x \,.$$

(65) $$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4(\omega\epsilon_1 - j\sigma_1)}{k_z} e^{-jk_x x} \, dk_x$$ [37]

(66) $$= 2(\omega\epsilon_1 - j\sigma_1) H_o^{(2)} (k_1 |x|).$$

The transform of $f_2(k_x)$ is

(67) $$F_2(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(k_x) e^{-jk_x x} \, dk_x$$

(68) $$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{ak_x^2} \sin^2\left(\frac{k_x a}{2}\right) e^{-jk_x x} \, dk_x$$

(69) $$F_2(x) = \begin{cases} \frac{1}{4a}(a - |x|) : & |x| \leq a \\ 0 & : |x| > a \end{cases} .$$

Then Parseval's Theorem gives

(70) $$\int_{-a/2}^{a/2} \overline{E} \times \overline{\Gamma} \cdot \hat{z} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_x) f_2^*(k_x) \, dk_x$$

$$= \int_{-\infty}^{\infty} F_1(x) F_2^*(x) dx = \frac{(\omega\epsilon_1 - j\sigma_1)}{2a} \int_{-a}^{a} (a - |x|)$$

$$\cdot H_o^{(2)} (k_1 |x|) dx .$$

[37] The integral in (65) is derived in:

R.T. Compton, Jr., "The Aperture Admittance of a Rectangular Waveguide Radiating into a Lossy Half-Space," Report 1691-1, Antenna Laboratory (in preparation).

Since the integrand is an even function of x, this may be written

$$(71) \quad \int_{-a/2}^{a/2} \overline{E} \times \overline{\Gamma} \cdot \hat{z}\, dx = \frac{\omega\epsilon_1 - j\sigma_1}{a} \int_0^a (a-x) H_0^{(2)}(k_1 x)\, dx .$$

The denominator in Eq. (37) is simply unity. Hence the admittance is given by:

$$(72) \qquad Y = \frac{\omega\epsilon_1 - j\sigma_1}{a} \int_0^a (a-x)\, H_0^{(2)}(k_1 x)\, dx.$$

It is convenient at this point to normalize (72) with respect to the free-space constants. Let

$$(73) \qquad k_o = \omega\sqrt{\mu_o \epsilon_o} = \frac{2\pi}{\lambda_o}$$

be the free-space propagation constant, where ϵ_o is the permittivity of free-space and λ_o is the free-space wavelength for the frequency ω. Also let

$$(74) \qquad Y_o = \sqrt{\frac{\epsilon_o}{\mu_o}}$$

be the characteristic admittance of free-space and define

$$(75) \qquad \eta = k_o x$$

$$(76) \qquad A = k_o a.$$

Then (72) may be written in the form

$$(77) \qquad Y = \frac{Y_o}{A}\left(\frac{k_1}{k_o}\right)^2 \int_o^A (A-\eta) H_o^{(2)}\left[\left(\frac{k_1}{k_o}\right)\eta\right] d\eta$$

$$= Y_o\left(\frac{k_1}{k_o}\right)^2 \int_o^A H_o^{(2)}\left[\left(\frac{k_1}{k_o}\right)\eta\right] d\eta$$

$$- \frac{Y_o}{A}\left(\frac{k_1}{k_o}\right)\left[\eta H_1^{(2)}\left(\frac{k_1}{k_o}\eta\right)\right]_o^A$$

where

$$\eta H_1^{(2)}\left(\frac{k_1}{k_o}\eta\right)\Bigg|_{\eta=0}$$

is understood to mean

$$(78) \qquad \eta H_1^{(2)}\left(\frac{k_1}{k_o}\eta\right)\Bigg|_{\eta=0} = \lim_{\eta\to 0^+}\left[\eta H_1^{(2)}\left(\frac{k_1}{k_o}\eta\right)\right].$$

Since

$$H_1^{(2)}(\rho) = J_1(\rho) - jN_1(\rho) \quad \text{and} \quad J_1(o) = 0,$$

$$(79) \qquad \lim_{\eta\to 0^+}\left[\eta H_1^{(2)}\left(\frac{k_1}{k_o}\eta\right)\right] = -j\lim_{\eta\to 0^+}\left[\eta N_1\left(\frac{k_1}{k_o}\eta\right)\right].$$

For small ρ,

$$(80) \qquad N_1(\rho) = -\frac{2}{\pi}\frac{1}{\rho}J_o(\rho) + \frac{2}{\pi}\ln\frac{\gamma\rho}{2}J_1(\rho)$$

$$-\frac{4}{\pi}\frac{\partial}{\partial\rho}\left[J_2(\rho) - \frac{1}{2}J_4(\phi) + \cdots\right]$$

where $\gamma = 1.781$, and thus

$$\lim_{\eta \to 0^+} \left[\eta H_1^{(2)}\left(\frac{k_1}{k_o}\eta\right)\right] = j\ \frac{2}{\pi}\ \frac{1}{k_1/k_o}\ . \tag{81}$$

Therefore (77) becomes:

$$Y = Y_o\left(\frac{k_1}{k_o}\right)^2 \int_o^A H_o^{(2)}\left[\left(\frac{k_1}{k_o}\right)\eta\right] d\eta - Y_o\left(\frac{k_1}{k_o}\right) H_1^{(2)}\left(\frac{k_1}{k_o}\ A\right) + j\ \frac{Y_o}{A}\ \frac{2}{\pi}\ . \tag{82}$$

Finally, with the substitution

$$\frac{k_1}{k_o} = C\, e^{-j\phi} \tag{83}$$

(82) becomes

$$Y = \frac{Y_o}{A}\, C A\, e^{-j2\phi} \int_o^{CA} H_o^{(2)}\left(\xi e^{-j\phi}\right) d\xi - \frac{Y_o}{A}\, CAe^{-j\phi}\ H_1^{(2)}\left(CAe^{-j\phi}\right) + j\ \frac{Y_o}{A}\ \frac{2}{\pi}\ . \tag{84}$$

As a check on the algebra, it may be seen that Y has the correct dimensions. A, C, and the integral are dimensionless. Hence Y has the dimensions of Y_o, i.e., mhos.

Let[38]

(85) $$IH_2(x, \alpha) = \int_0^x H_0^{(2)}(\xi e^{+j\alpha})d\xi.$$

Then Y, normalized to $\frac{Y_0}{A}$, is given by

(86) $$\frac{A}{Y_0} Y = CAe^{-j\phi}\left[e^{-j\phi} IH_2(CA, -\phi) - H_1^{(2)}(CAe^{-j\phi})\right] + j\,\frac{2}{\pi} \quad .$$

---

[38] The function $IH_2(x, \alpha)$, and also the integrals

$$IJ(x, \alpha) = \int_0^x J_0(\xi e^{+j\alpha})d\xi$$

$$IN(x, \alpha) = \int_0^x N_0(\xi e^{-j\alpha})d\xi$$

$$IH_1(x, \alpha) = \int_0^x H_0^{(1)}(\xi e^{+j\alpha})d\xi$$

have been tabulated by the author for $0 \leq x \leq 10.0$, $-90° \leq \phi \leq 90°$, and will be published in a forthcoming Antenna Laboratory report.

The values of $H_1^{(2)}(z)$ for complex z were obtained from the following two tables:

(a) "Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments," Mathematical Tables Project, National Bureau of Standards, Columbia University Press, New York, 1943.

(b) "Table of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for Complex Arguments," National Bureau of Standards, Columbia University Press, New York, 1950.

Equation (86) has been evaluated for $0 \leq CA \leq 10$ and $0° \leq \theta \leq 90°$, and is shown in Fig. 3. It is convenient to plot the normalized quantity $\frac{A}{Y_o} Y$, since then only one curve need be drawn for all apertures.

For a fixed aperture size and fixed frequency, Fig. 3 shows the behavior of the admittance Y as a function of C, and hence as a function of ϵ_1 and σ_1.

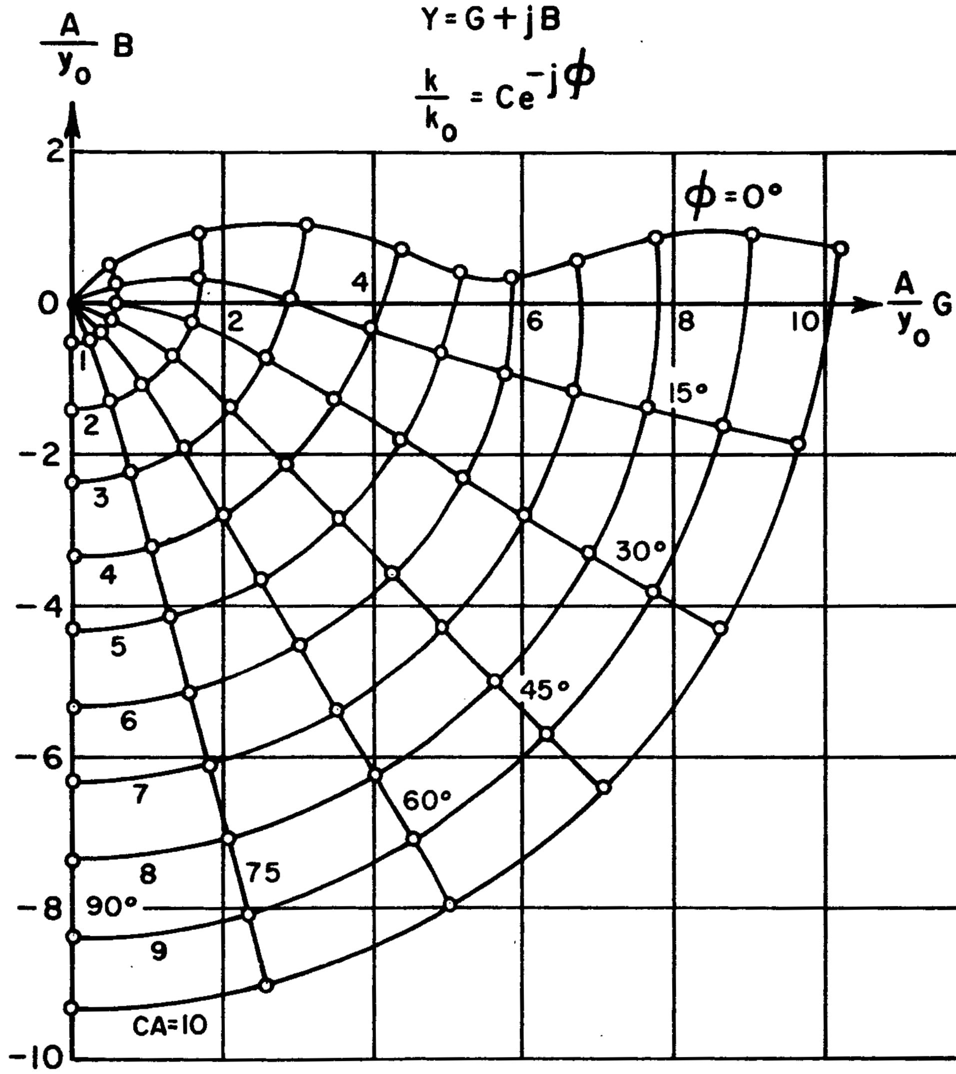

Fig. 3--The normalized admittance $\frac{A}{Y_o}$ Y for an infinite medium

If one is interested in the admittance of the slot as a function of the slot dimension $A(=k_o a)$, the shape of the curves in Fig. 3 is misleading because A is included in the normalizing constant for Y. This difficulty may be remedied by using the data of Fig. 3 to determine the quantity $\frac{Y}{Y_o}$ as a function of A (and ϕ) for fixed C.

As a check on the numerical results shown in Fig. 3, it is helpful to consider two limiting cases. First, suppose the semi-infinite region has $\epsilon_1 = \sigma_1 = 0$ so that $C = 0$. (The point given by $CA = 0$ in Fig. 2 also corresponds to the d.c. case, i.e., $\omega = 0$, but the interpretation near $\omega = 0$ is somewhat tricky. This case is discussed below.) Then it is clear from Eq. (59) that $H_y(x, y, o)=0$ and therefore $Y = 0$. This accounts for the fact that the curves in Fig. 3 approach the origin as $C \to 0$.

Second, consider the case where C is large (and $\phi \neq 0$). Since the function $H_o^{(2)}\left(\frac{k_1}{k_o}\eta\right)$ decays rapidly to zero for complex $\frac{k_1}{k_o}$ as η becomes large, the integral in (84) may be replaced by

$$\int_o^{CA} H_o^{(2)}(\xi e^{-j\phi})\, d\xi \simeq \int_o^{\infty} H_o^{(2)}(\xi e^{-j\phi}) d\xi \tag{87}$$

with little change in value. From the Fourier Transform pair:[39]

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[39]See Reference 37.

$$\int_{-\infty}^{\infty} H_o^{(2)}(\alpha|x|)e^{+jk_x x}\,dx = \frac{2}{\sqrt{\alpha^2 - k_x^2}} \tag{88}$$

where

$$\mathrm{Re}[\sqrt{\alpha^2 - k_x^2}] \geq 0 \tag{89}$$

$$\mathrm{Im}[\sqrt{\alpha^2 - k_x^2}] \leq 0 \tag{90}$$

it is seen that

$$\int_{o}^{\infty} H_o^{(2)}(\alpha x)dx = \frac{1}{\alpha} \quad . \tag{91}$$

Hence the integral in (87) may be approximated by

$$\int_{o}^{CA} H_o^{(2)}(\xi e^{-j\phi})d\xi \simeq e^{j\phi} . \tag{92}$$

Also for large C,

$$(CA)\, H_1^{(2)}(CA\, e^{-j\phi}) \cong 0 \quad (\phi \neq 0) \quad . \tag{93}$$

Making use of these gives the following approximate form for (84):

$$Y \cong \frac{Y_o}{A} CA\, e^{-j\phi} + j\, \frac{Y_o}{A}\, \frac{2}{\pi} \tag{94}$$

or

$$\frac{A}{Y_o} Y \cong CA\, e^{-j\phi} + j\, \frac{2}{\pi} \quad . \tag{95}$$

This behavior is clearly indicated in Fig. 3. Comparison of Eq. (95) with the curves in Fig. 3 shows that (95) is quite accurate for $\phi > 15°$ and $CA > 7$.

As mentioned above, the behavior of Y as a function of frequency is somewhat tricky as $\omega \to 0$. From (86)

$$Y = Y_o C\, e^{-j\phi} \left[e^{-j\phi} IH_2(CA, -\phi) - H_1^{(2)}(CA\, e^{-j\phi}) \right] + j\, \frac{2}{\pi}\, \frac{Y_o}{A} \quad . \tag{96}$$

For low frequencies, if $\sigma_1 \neq 0$, C and CA are given by:

$$C = \left| \frac{k_1}{k_o} \right| = \left| \frac{\sqrt{-j\omega\mu_o(j\omega\epsilon_1 + \sigma_1)}}{\sqrt{\omega\ \mu_o \epsilon_o}} \right| = \sqrt{\frac{\sigma_1}{\omega\epsilon_o}} \tag{97}$$

$$CA = \left| \frac{k_1}{k_o} \right| k_o a = \sqrt{\omega\mu_o\sigma_1}\ a \quad . \tag{98}$$

Hence $CA \to 0$ as $\omega \to 0$. For small values of ρ,

$$H_o^{(2)}(\rho) \cong 1 - j\, \frac{2}{\pi}\, \ln \frac{\gamma\rho}{2} \quad . \tag{99}$$

Therefore putting $\rho = \xi e^{j\alpha}$ and substituting (99) in (85) gives for small x

$$\text{(100)} \qquad IH_2(x,\alpha) \cong x\left(1 + j\,\frac{2}{\pi} - j\,\frac{2}{\pi}\,\ln\frac{\gamma x}{2} + \frac{2}{\pi}\,\alpha\right)$$

($\gamma = 1.781$). Similarly, for small ρ,

$$\text{(101)} \qquad H_1^{(2)}(\rho) \cong j\,\frac{2}{\pi}\,\frac{1}{\rho} - j\,\frac{\rho}{\pi}\,\ln\frac{\gamma\rho}{2} \ .$$

Using (100) and (101) in (96) gives

$$\text{(102)} \qquad Y = Y_o C\,e^{-j\phi}\left[e^{-j\phi}\,CA\left(1 + j\,\frac{2}{\pi} - j\,\frac{2}{\pi}\,\ln\frac{\gamma CA}{2} - \frac{2}{\pi}\,\phi\right) + j\,\frac{CAe^{-j\phi}}{\pi}\,\ln\frac{\gamma CA}{2} + \frac{CAe^{-j\phi}}{\pi}\,\phi\right] .$$

Substituting (97) and (98) and collecting terms gives for the leading term in

$$\text{(103)} \qquad Y \cong -j\,\frac{\sigma_1 a}{\pi}\,e^{-j2\phi}\,\ln\frac{\gamma\sqrt{\omega\mu_o\sigma_1}\;a}{2} \ .$$

Since

$$\text{(104)} \qquad -\phi = \arg\sqrt{-j\omega\mu_o(j\omega\epsilon_1 + \sigma_1)}$$

for small ω,

$$\text{(105)} \qquad \phi \cong \frac{\pi}{4}$$

and therefore

$$\text{(106)} \qquad Y \cong -\frac{\sigma_1 a}{\pi}\,\ln\frac{\gamma\sqrt{\omega\mu_o\sigma_1}\;a}{2} \ .$$

Because

$$\lim_{\omega \to 0^+} \ln \frac{\gamma \sqrt{\omega \mu_o \sigma_1}\, a}{2} = -\infty \tag{107}$$

it is seen that $Y \to +\infty$ as $\omega \to 0$, for any $\sigma_1 \neq 0$.

If $\sigma_1 = 0$, however, instead of (97) and (98) we use

$$C = \left| \frac{k_1}{k_o} \right| = \sqrt{\frac{\epsilon_1}{\epsilon_o}} \tag{108}$$

$$CA = \omega \sqrt{\mu_o \epsilon_1}\, a \tag{109}$$

in (102). This gives

$$Y = \omega \epsilon_1 a \left[1 + j \, \frac{1}{\pi} \left(2 - \ln \frac{\gamma \omega \sqrt{\mu_o \epsilon_1}\, a}{2} \right) \right] \tag{110}$$

which is an interesting result because

$$\lim_{\omega \to 0} Y = \begin{cases} +\infty: & \sigma_1 \neq 0 \\ 0: & \sigma_1 = 0 \end{cases} \quad . \tag{111}$$

This peculiar behavior may be understood by examining carefully the logarithmic term in (102). For any non-zero conductivity, this term contributes a singularity at $\omega = 0$. The lower the conductivity, the lower the frequency must be before this term contributes appreciably to Y. In the limit, the singularity at $\omega = 0$ disappears.

Finally, it is interesting to make the following observation. Suppose the parallel-plate transmission line shown in Fig. 2, instead of feeding a semi-infinite half-space, feeds an infinite section of transmission line with the same dimensions and with a lossy dielectric between the plates, as shown in Fig. 4. The characteristics admittance of the line for $z > 0$ is

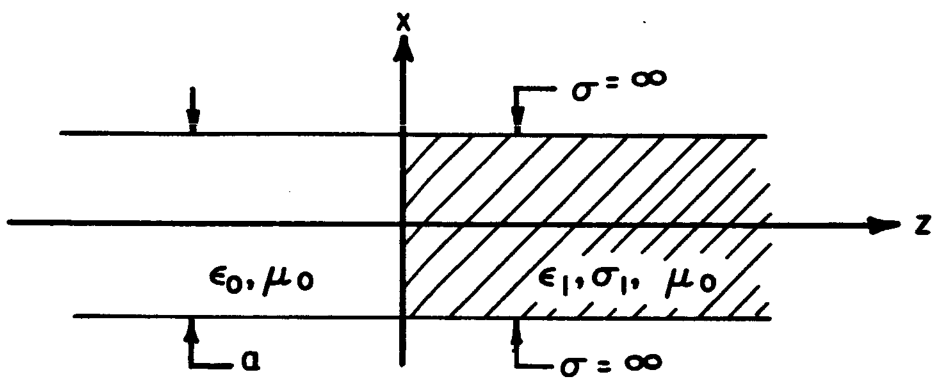

Fig. 4--Infinite transmission line model

(112) $$Y_C = \sqrt{\frac{j\omega\epsilon_1 + \sigma_1}{j\omega\mu_o}} \quad .$$

The terminating admittance, Y', for the section of line $z < 0$ is simply Y_C. Hence, after some algebra

(113) $$\frac{A}{Y_o} Y' = \frac{A}{Y_o}\sqrt{\frac{j\omega\epsilon_1 + \sigma_1}{j\omega\mu_o}} = CA\,e^{-j\phi}$$

which is the first term of (95). Thus, except for a constant $j\frac{2}{\pi}$, the admittance of the slot is correctly given by the model in Fig. 4, for large CA.

B. The Lossy Slab

Next the case where the infinite slot radiates into a lossy slab of finite thickness will be considered. The geometry is shown in Fig. 5. As before, the slot has width "a" and the electric field in the slot is given by Eq. (36). The slab has thickness "d" and propagation constant k_1. In the region $z > d$, the medium is free-space, with propagation constant k_o.

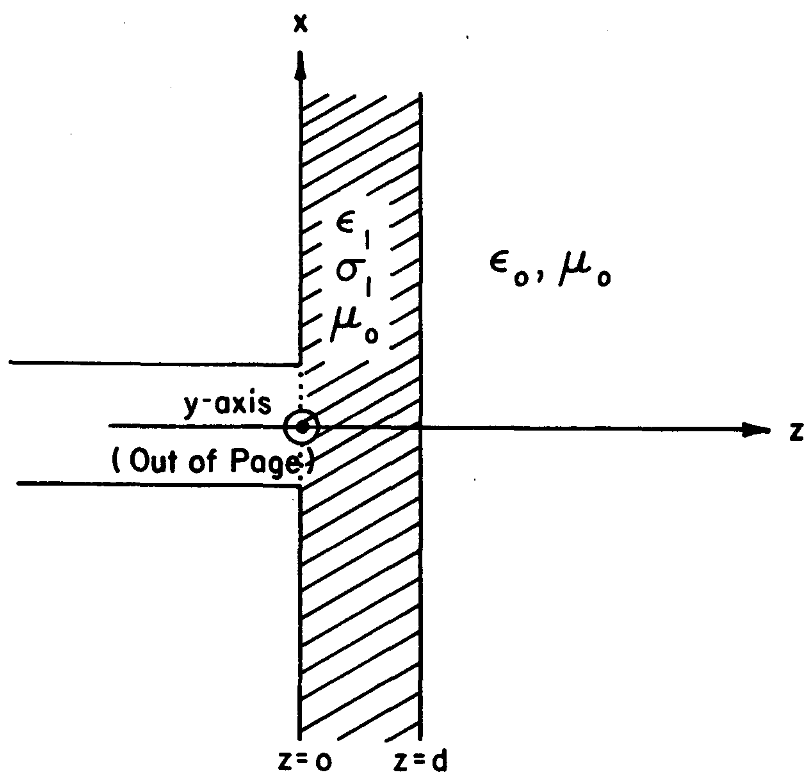

Fig. 5--Slot radiating through finite slab

As in Part A, the fields are TE to the y-axis and may be derived from a vector potential of the form given by Eq. (40). The electromagnetic fields are then given by Eqs. (42) and (43), with the appropriate value of k used for each region of space.

In the slab, which will be called Medium 1, ψ will consist of both an "incident" and a "reflected" component:

$$(114) \qquad \psi_1(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [I(k_x)e^{-jk_{z1}z} + R(k_x)e^{+jk_{z1}}] \cdot e^{-jk_x x}\, dk_x \,.$$

In the free-space region, Medium 0, ψ has only a transmitted component,

$$(115) \qquad \psi_o(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(k_x)e^{-jk_{zo}z}\, e^{-jk_x x}\, dk_x \,.$$

In (114) and 115) k_{z1} and k_{zo} are the z-direction propagation constants given by

$$(116) \qquad k_{z1} = \sqrt{k_1^2 - k_x^2}$$

$$(117) \qquad k_{zo} = \sqrt{k_o^2 - k_x^2}$$

which are chosen so that

$$(118) \qquad \mathrm{Re}(k_{z1}),\ \mathrm{Re}(k_{zo}) \geq 0$$

(119) $\quad \mathrm{Im}(k_{z1}),\ \mathrm{Im}(k_{zo}) \leq 0$.

Applying Eqs. (42) and (43) gives for the fields:

(120) $$E_{x1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [-jk_{z1} I(k_x) e^{-jk_{z1} z} + jk_{z1} R(k_x) e^{+jk_{z1} z}] \cdot e^{-jk_x x}\, dk_x$$

(121) $$H_{y1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -(j\omega\epsilon_1 + \sigma_1)[I(k_x) e^{-jk_{z1} z} + R(k_x) e^{+jk_{z1} z}] \cdot e^{-jk_x x}\, dk_x$$

(122) $$E_{xo}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -jk_{zo} T(k_x) e^{-jk_{zo} z}\, e^{-jk_x x}\, dk_x$$

(123) $$H_{yo}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -j\omega\epsilon_o T(k_x) e^{-jk_{zo} z}\, e^{-jk_x x}\, dk_x .$$

For $z = 0$, (120) gives the relation

(124) $$E_{x1}(x, y, o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} jk_{z1} [-I(k_x) + R(k_x)] e^{-jk_x x}\, dk_x .$$

Taking the inverse transform of (124) and making use of (53) gives

(125) $$jk_{z1}[-I(k_x) + R(k_x)] = \frac{2}{\sqrt{a}\, k_x} \sin\left(\frac{k_x a}{2}\right).$$

Also, applying boundary conditions at $z = d$:

(126) $$E_{x1}(x, d) = E_{xo}(x, d)$$

(127) $$H_{y1}(x, d) = H_{yo}(x, d)$$

gives the two relations

(128) $$k_{z1}\left[I(k_x)e^{-jk_{z1}d} - R(k_x)e^{+jk_{z1}d}\right] = k_{zo}T(k_x)e^{-jk_{zo}d}$$

(129) $$(j\omega\epsilon_1 + \sigma_1)\left[I(k_x)e^{-jk_{z1}d} + R(k_x)e^{+jk_{z1}d}\right] = j\omega\epsilon_o T(k_x)e^{-jk_{zo}d} .$$

Solving (125), (128), and (129) simultaneously yields for $I(k_x)$ and $R(k_x)$

(130) $$I(k_x) = \frac{\left(\frac{k_{z1}}{k_{zo}} + \frac{k_1^2}{k_o^2}\right)e^{+jk_{z1}d}}{\frac{k_1^2}{k_o^2}\cos k_{z1}d + j\frac{k_{z1}}{k_{zo}}\sin k_{z1}d} \frac{j}{k_x k_z\sqrt{a}} \sin\left(\frac{k_x a}{2}\right)$$

(131) $$R(k_x) = \frac{\left(\frac{k_{z1}}{k_{zo}} - \frac{k_1^2}{k_o^2}\right)e^{-jk_{z1}d}}{\frac{k_1^2}{k_o^2}\cos k_{z1}d + j\frac{k_{z1}}{k_{zo}}\sin k_{z1}d} \frac{j}{k_x k_z\sqrt{a}} \sin\left(\frac{k_x a}{2}\right) .$$

Equations (130) and (131) along with (120) and (121) determine the fields in Region 1.

Now applying Eqs. (37), (39), and (60) and making use of Parseval's theorem yields for the admittance

$$(132) \qquad Y = \frac{1}{2\pi a}\int_{-\infty}^{\infty} \frac{1+jC(k_x)\tan k_{z1}d}{C(k_x)+j\tan k_{z1}d}\ \frac{4k_1^2}{\omega\mu_o k_x^2 k_{z1}} \sin^2\left(\frac{k_x a}{a}\right) dk_x$$

where

$$(133) \qquad C(k_x) = \frac{k_1^2}{k_o^2}\ \frac{k_{zo}}{k_{z1}} \quad .$$

The integral in (132) is difficult to evaluate analytically. To obtain quantitative results from (132), it was necessary to resort to numerical integration techniques.

Before discussing these results, however, it is interesting to make the following observation.

Let

$$(134) \qquad \beta = e^{-jk_{z1}d} \quad .$$

Then

$$(135) \qquad \tan k_{z1}d = -j\,\frac{\frac{1}{\beta} - \beta}{\frac{1}{\beta} + \beta} = -j\,\frac{1-\beta^2}{1+\beta^2} \quad ,$$

and

$$(136) \qquad \frac{1+jC(k_x)\tan k_{z1}d}{C(k_x)+j\tan k_{z1}d} = \frac{1+C(k)\,\frac{1-\beta^2}{1+\beta^2}}{C(k_x)+\frac{1-\beta^2}{1+\beta^2}} = \frac{D(k_x)-\beta^2}{D(k_x)+\beta^2} \quad ,$$

where

$$(137) \qquad D(k_x) = \frac{C(k_x) + 1}{C(k_x) - 1} = \frac{k_1^2 k_{zo} + k_o^2 k_{z1}}{k_1^2 k_{zo} - k_o^2 k_{z1}} \quad .$$

Equation (136) may be expanded

$$(138) \qquad \frac{D(k_x) - \beta^2}{D(k_x) + \beta^2} = \frac{1 - \frac{\beta^2}{D(k_x)}}{1 + \frac{\beta^2}{D(k_x)}} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{\beta^2}{D(k_x)} \right)^n ,$$

which is valid if

$$(139) \qquad \left| \frac{\beta^2}{D(k_x)} \right| < 1 \ .$$

Then, substituting for β^2 and $D(k_x)$ in (138) yields the expansion

$$(140) \qquad \frac{1 + jC(k_x) \tan k_{z1} d}{C(k_x) + j \tan k_{z1} d} = 1 + 2 \sum_{n=1}^{\infty} \left(\frac{k_o^2 k_{z1} - k_1^2 k_{zo}}{k_o^2 k_{z1} + k_1^2 k_{zo}} \right)^n e^{-j2nk_{z1}d} \quad .$$

This result is interesting because the terms of the series in (140) are easily interpreted. For a thick, lossy slab, $e^{-j2nk_{z1}d}$ is small and the terms of the series are negligible in comparison with the leading term of 1. If the substitution

$$(141) \qquad \frac{1 + j C(k_x) \tan k_{z1} d}{C(k_x) + j \tan k_{z1} d} \cong 1$$

is made in (132) the resulting admittance is the same as in part A above, i.e., the admittance of a slot in an infinite lossy region. The exponential $e^{-j2nk_{z1}d}$ represents the phase shift and attenuation undergone by a wave propagating from the aperture out to $z = d$ and back to the aperture. Also, $\left(\frac{k_o^2 k_{z1} - k_1^2 k_{zo}}{k_o^2 k_{z1} + k_1^2 k_{zo}}\right)$ may be shown to be the reflection coefficient associated with a reflection at $z = d$. Hence, the n-th term in the series (140) represents the effect on the admittance of the wave arriving at the aperture after n reflections between $z = 0$ and $z = d$.[40]

Although the expansion in (140) does not help to evaluate (132), it serves as a check on the derivation of (132) and allows a direct physical meaning to be attached to the integrand.

Now we return to the evaluation of (132). First, the integral may be written in terms of normalized parameters. Let

(142) $$\rho = C e^{-j\phi} = \frac{k_1}{k_o}$$

(143) $$\eta = \frac{k_x}{k_o}$$

(144) $$A = k_o a$$

and

(145) $$D = k_o d.$$

[40] B. van der Pol and H. Bremmer, Phil. Mag. (7)24(1937)825.

Then (132) becomes

$$(146) \qquad Y = \frac{2}{\pi}\,\frac{Y_o}{A}\int_{-\infty}^{\infty}\frac{1+j\rho^2\dfrac{\sqrt{1-\eta^2}}{\sqrt{\rho^2-\eta^2}}\tan\sqrt{\rho^2-\eta^2}\,D}{\rho^2\dfrac{\sqrt{1-\eta^2}}{\sqrt{\rho^2-\eta^2}}+j\tan\sqrt{\rho^2-\eta^2}\,D}\;\frac{\rho^2\sin^2\left(\dfrac{\eta A}{2}\right)}{\eta^2\sqrt{\rho^2-\eta^2}}\,d\eta .$$

Equation (146) has been evaluated numerically on the Ohio State University IBM 7094 digital computer for various values of ρ, A, and D. Since the integrand in (146) is an even function of η, Y may be found by integrating over the range $0 \leq \eta \leq \infty$ and multiplying the result by 2. The integral over the finite range $0 \leq \eta \leq \eta_o$ is actually used to approximate the integral over $0 \leq \eta \leq \infty$, where η_o is chosen large enough to include the region where the integrand is significantly different from zero. η_o is chosen as follows. For large η,

$$(147) \qquad \tan\sqrt{\rho^2-\eta^2}\,D \cong \tan(-j\eta D) \cong -j$$

and hence,

$$(148) \qquad \frac{1+j\rho^2\dfrac{\sqrt{1-\eta^2}}{\sqrt{\rho^2-\eta^2}}\tan\sqrt{\rho^2-\eta^2}\,D}{\rho^2\dfrac{\sqrt{1-\eta^2}}{\sqrt{\rho^2-\eta^2}}+j\tan\sqrt{\rho^2-\eta^2}\,D} \cong 1 .$$

Let

(149) $\eta_1 = 10|\rho| = 10C,$

(150) $\eta_2 = 3/D,$

and

(151) $\eta_o = \max(\eta_1, \eta_2).$

Then for $|\eta| > |\eta_o|$, Eqs. (147) and (148) hold to a good approximation. Now from (146), the normalized admittance $\frac{A}{Y_o}Y$ may be written,

$$(152) \quad \frac{A}{Y_o} Y = \frac{4}{\pi}\int_{-\infty}^{\infty} \frac{1+\rho^2 \frac{\sqrt{1-\eta^2}}{\sqrt{\rho^2-\eta^2}} \tan\sqrt{\rho^2-\eta^2}\, D}{\rho^2\frac{\sqrt{1-\eta^2}}{\sqrt{\rho^2-\eta^2}} + j\tan\sqrt{\rho^2-\eta^2}\, D} \; \frac{\rho^2 \sin^2\left(\frac{\eta A}{2}\right)}{\eta^2\sqrt{\rho^2-\eta^2}}\, d\eta$$

$$+ \frac{4}{\pi}\int_{-\infty}^{\infty} \frac{\rho^2 \sin^2\left(\frac{\eta A}{2}\right)}{\eta^2\sqrt{\rho^2-\eta^2}}\, d\eta \, .$$

The following relations then hold

$$(153)^{41} \quad \left|\frac{4}{\pi}\int_{-\infty}^{\infty} \frac{\rho^2 \sin^2\left(\frac{\eta A}{2}\right)}{\eta^2\sqrt{\rho^2-\eta^2}}\, d\eta\right| \le \left|\frac{4}{\pi}\int_{\eta_o}^{\infty} \frac{\rho^2\, d\eta}{\eta^2\sqrt{\rho^2-\eta^2}}\right|$$

---

[41] Of course, to say $a \le b \cong c$ does not actually "prove" anything about the relation between a and c. However, the argument can easily be made rigorous.

(153) cont. $$\cong \frac{4}{\pi} C^2 \int_{\eta_o}^{\infty} \frac{d\eta}{\eta^3} = \frac{2}{\pi}\left(\frac{C}{\eta_o}\right)^2 ,$$

and therefore the condition

(154) $$\eta_o \geq 10\ C$$

implies

(155) $$\left| \frac{4}{\pi} \int_{\eta_o}^{\infty} \frac{\rho^2 \sin^2\left(\frac{\eta A}{2}\right)}{\eta^2 \sqrt{\rho^2 - \eta^2}} d\eta \right| \leq \frac{2}{100\,\pi} < 0.01 .$$

Hence to within this error

(156) $$\frac{A}{Y_o} Y = \frac{4}{\pi} \int_{o}^{\eta_o} \frac{1 + j\frac{\sqrt{1-\eta^2}}{\sqrt{\rho^2-\eta^2}} \tan\sqrt{\rho^2-\eta^2}\, D}{\rho^2 \frac{\sqrt{1-\eta^2}}{\sqrt{\rho^2-\eta^2}} + j \tan\sqrt{\rho^2-\eta^2}\, D} \; \frac{\rho^2 \sin^2\left(\frac{\eta A}{2}\right)}{\eta^2\sqrt{\rho^2-\eta^2}} d\eta .$$

From the results obtained above in Part A, an error less than 0.01 seems acceptable.

The integral in (156) was evaluated by means of Simpson's rule. Several diffierent increment sizes were chosen throughout the range $0 \leq \eta \leq \eta_o$. For some values of η, the integrand in (156) fluctuates rapidly, while for other values of η it is slowly varying. To reduce the computer running time, it was desirable to make the increments large in the regions where the integrand is slowly varying.

The results of this calculation are shown in Fig. 6 through Fig. 23. These figures show the normalized admittance $\frac{A}{Y_o} Y$ in terms of normalized conductance and susceptance,

$$\frac{A}{Y_o} Y = \frac{A}{Y_o} G + j \frac{A}{Y_o} B \tag{157}$$

for nine combinations of A and D. Although there is perhaps no need to include "A" in the normalization of Y (since it has a given value for each curve), it has been included so these curves may be compared directly with Fig. 2, for the infinite lossy medium.

Figures 6 through 11 show the admittance for A = 1. Figures 6 and 7 and for D = 1, Figs. 8 and 9 for D = 0.5, and Figs. 10 and 11 for D = 0.25. Figures 12 through 17 are for A = 0.5 with D = 1, 0.5, and 0.25. Figures 18 through 23 show the results for A = 0.25 and again with D = 1, 0.5, and 0.25.

For the case where the slab has large loss (k has a large imaginary part), the admittance is seen to be the same as in Part A above. This result is expected, of course, because for large loss the effect of reflections at z = d should be negligible at the aperture. On the other hand, for a low loss medium, the admittance is found to oscillate rapidly as a function of k_1/k_o. In fact, for $\phi = 0°$, the admittance cannot be shown meaningful on the same graph as for $\phi \geq 15°$, because it oscillates too violently. For this reason for

$\phi = 0°$ the admittance has been plotted on a separate curve for each combination of A and D.

It is interesting to note that the curves of admittance are in some cases double-valued. That is, the same admittance can result from two or more values of k. This is evidenced by the folding over in some of the curves (for example, Fig. 8).

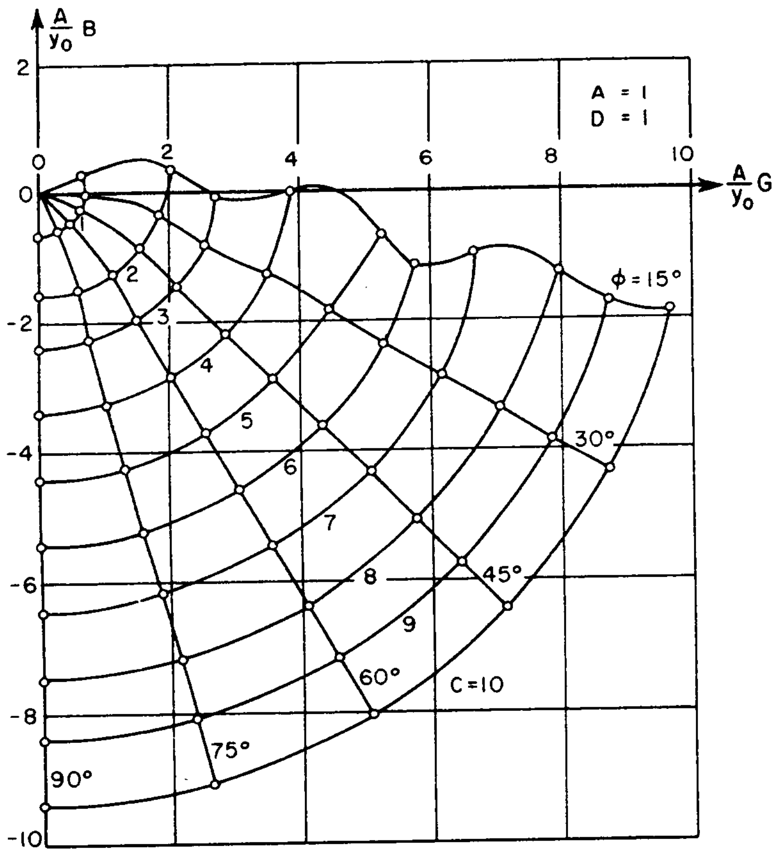

Fig. 6--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

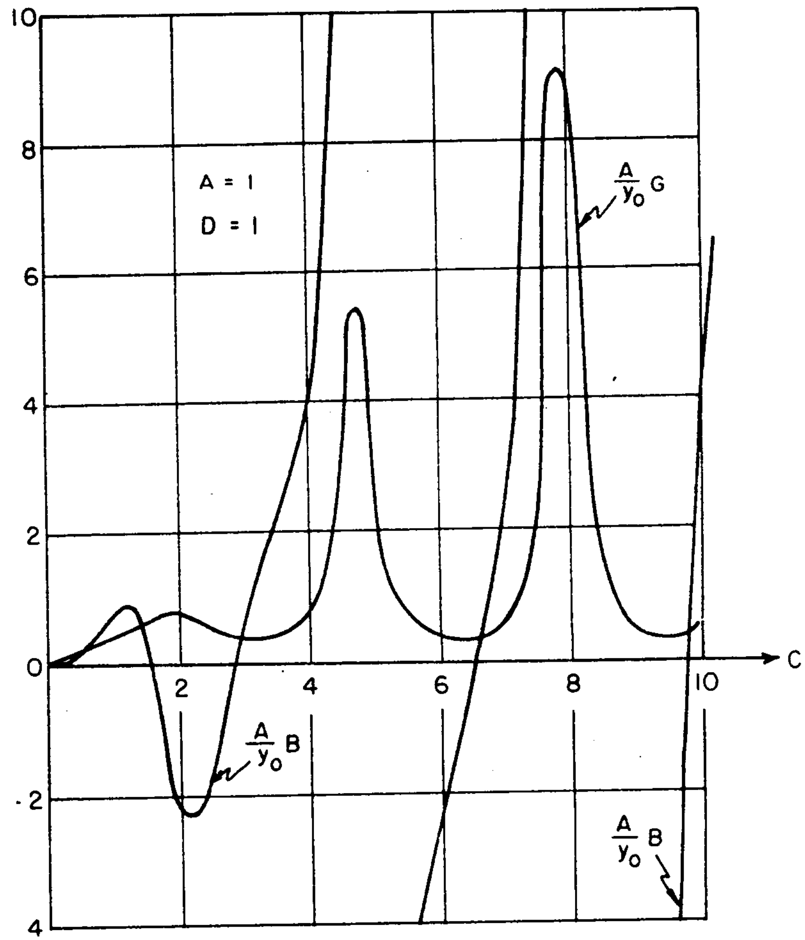

Fig. 7--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

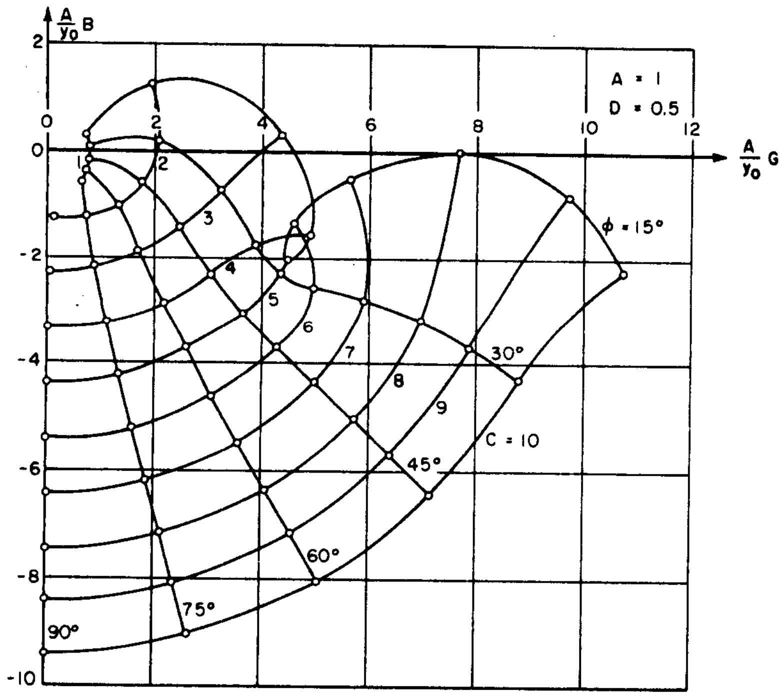

Fig. 8--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

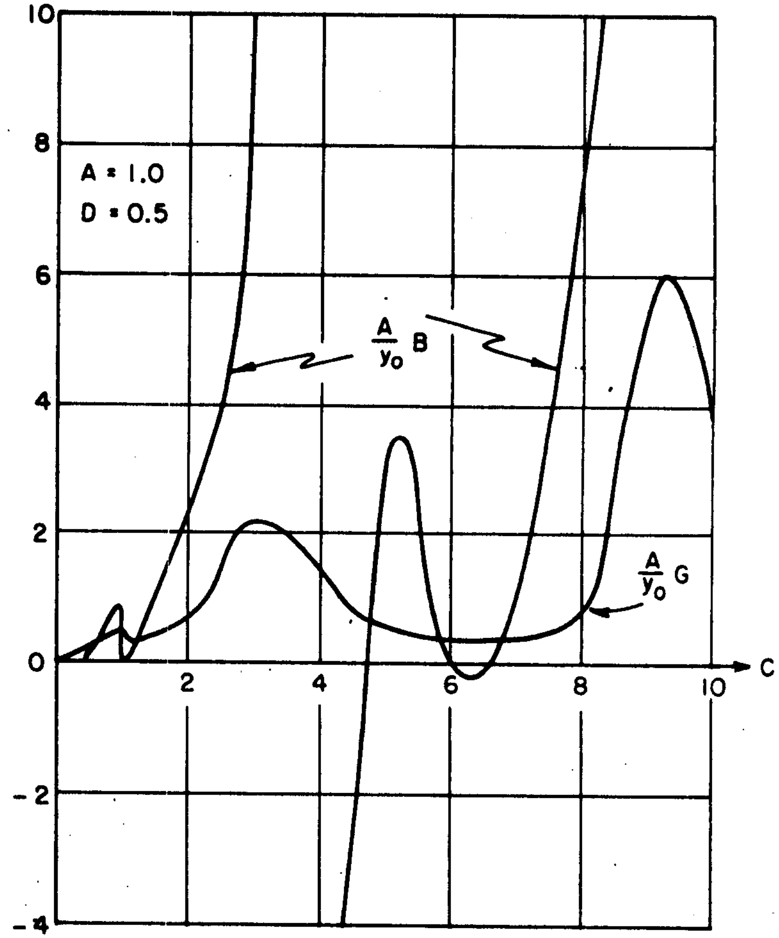

Fig. 9--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

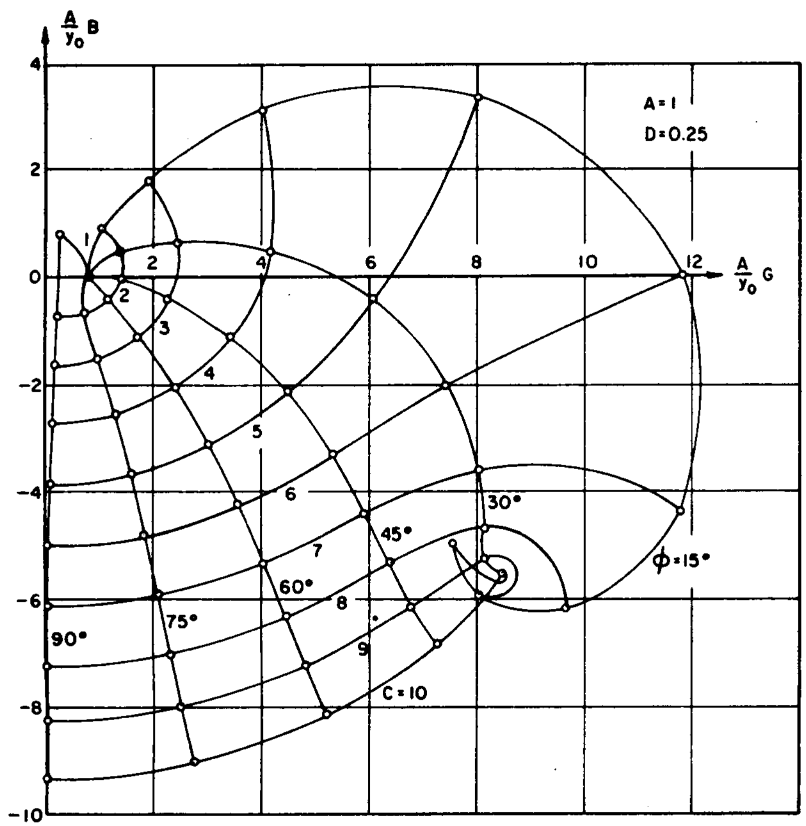

Fig. 10--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

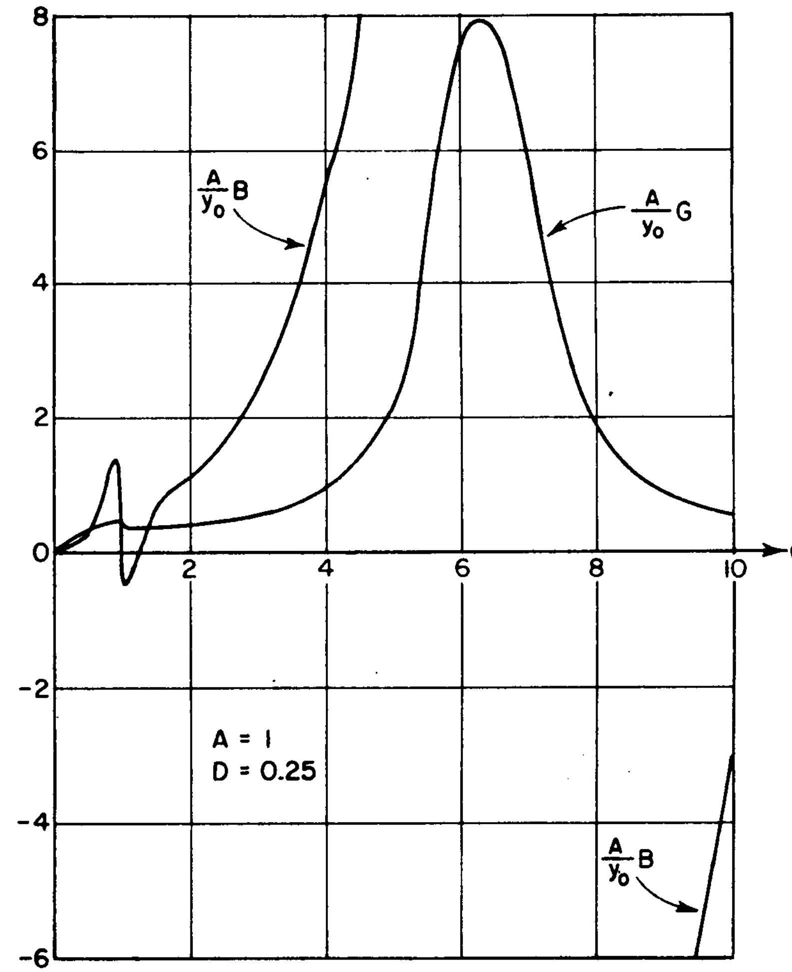

Fig. 11--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

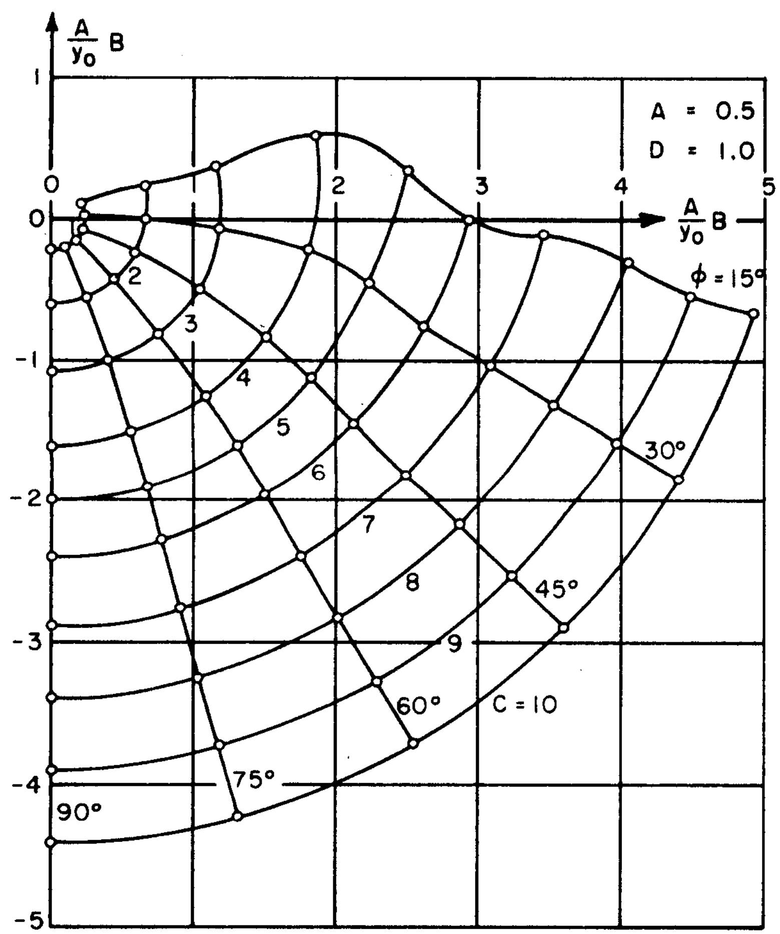

Fig. 12--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

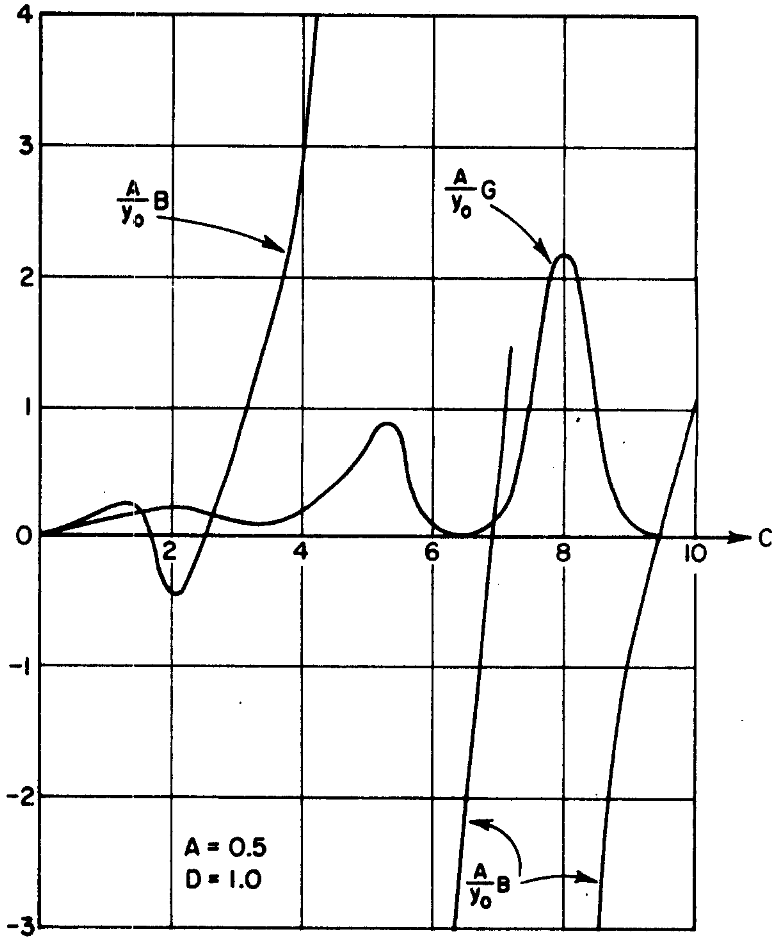

Fig. 13--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

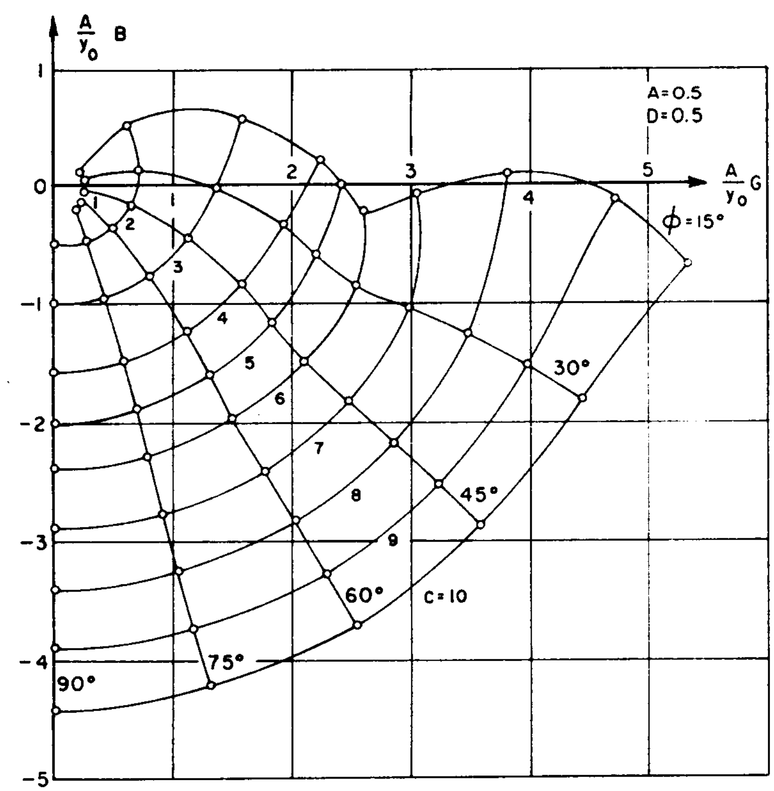

Fig. 14--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

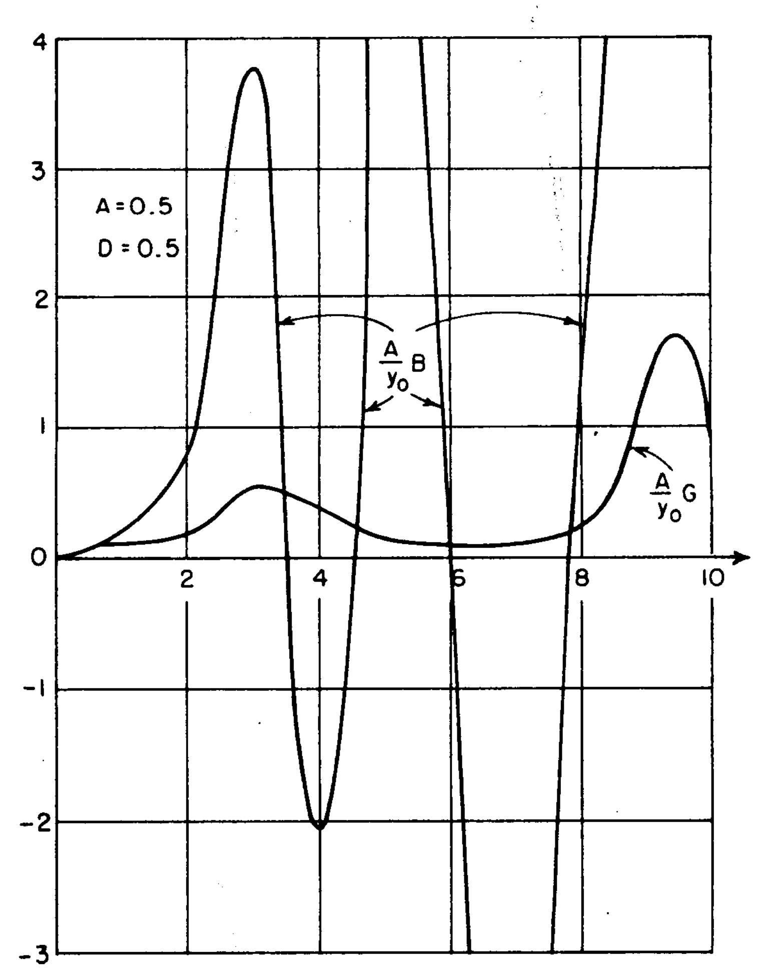

Fig. 15--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

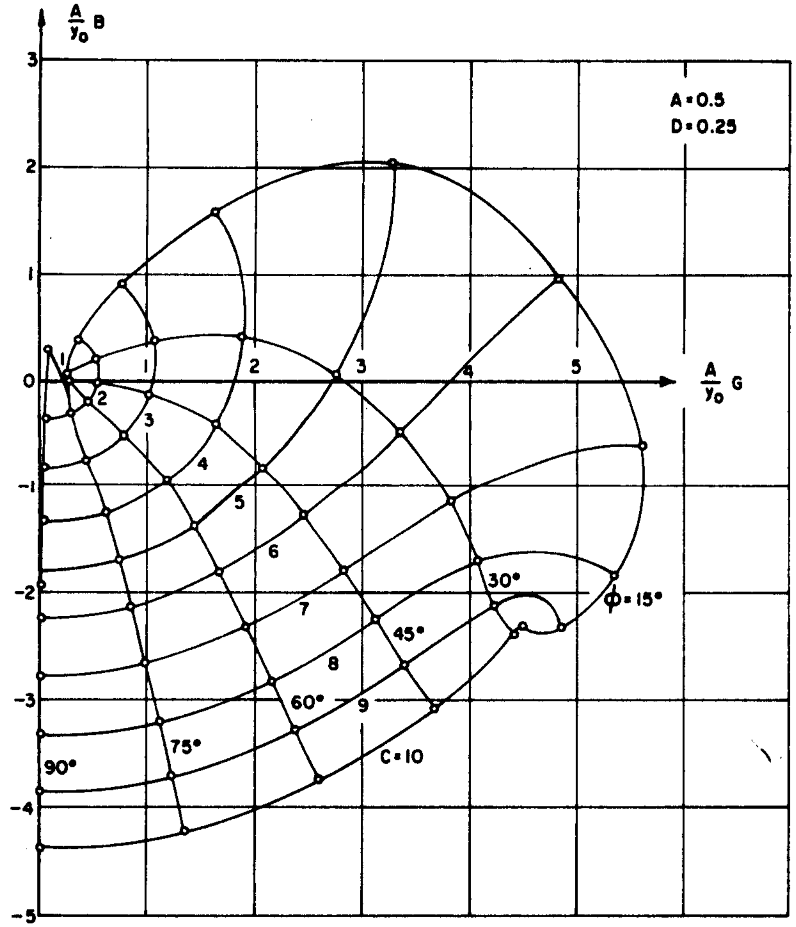

Fig. 16--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

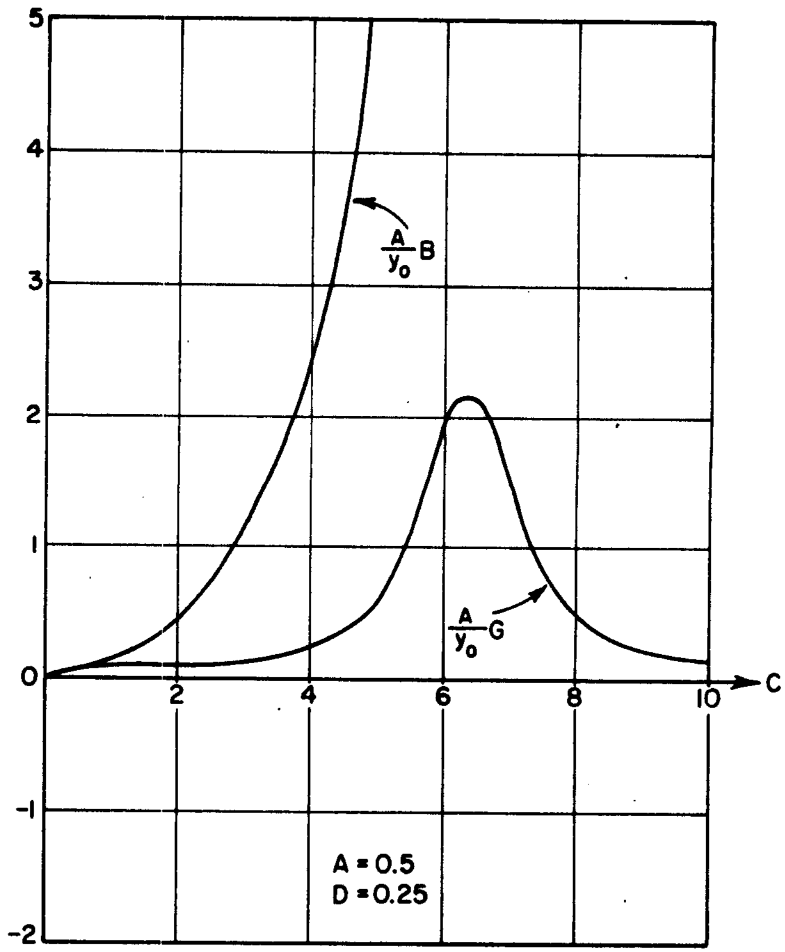

Fig. 17--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

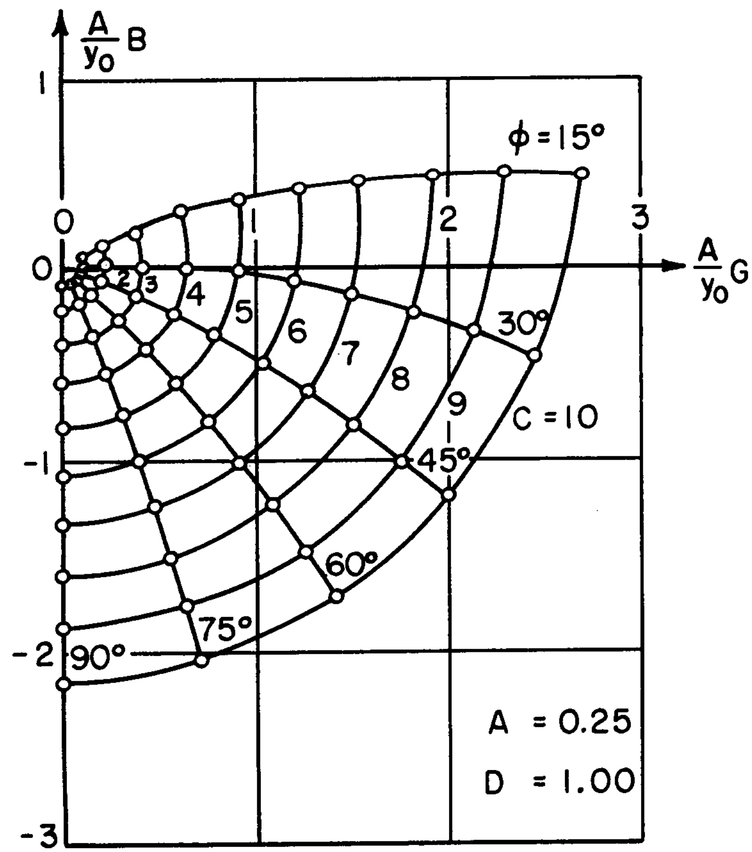

Fig. 18--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

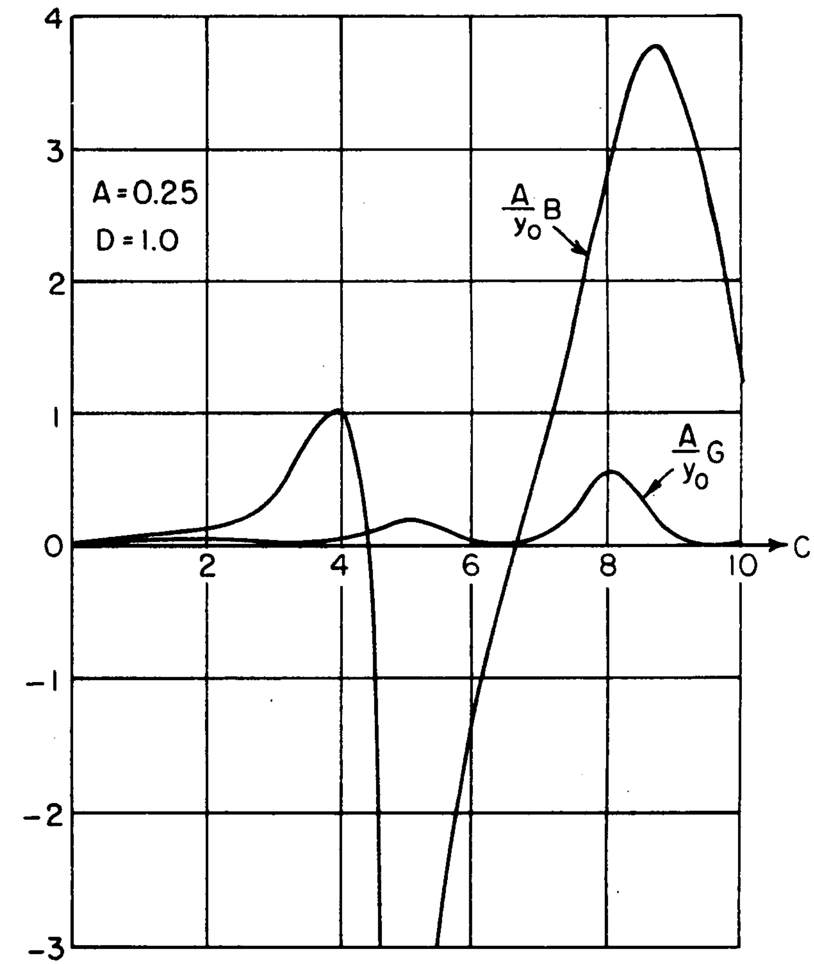

Fig. 19--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

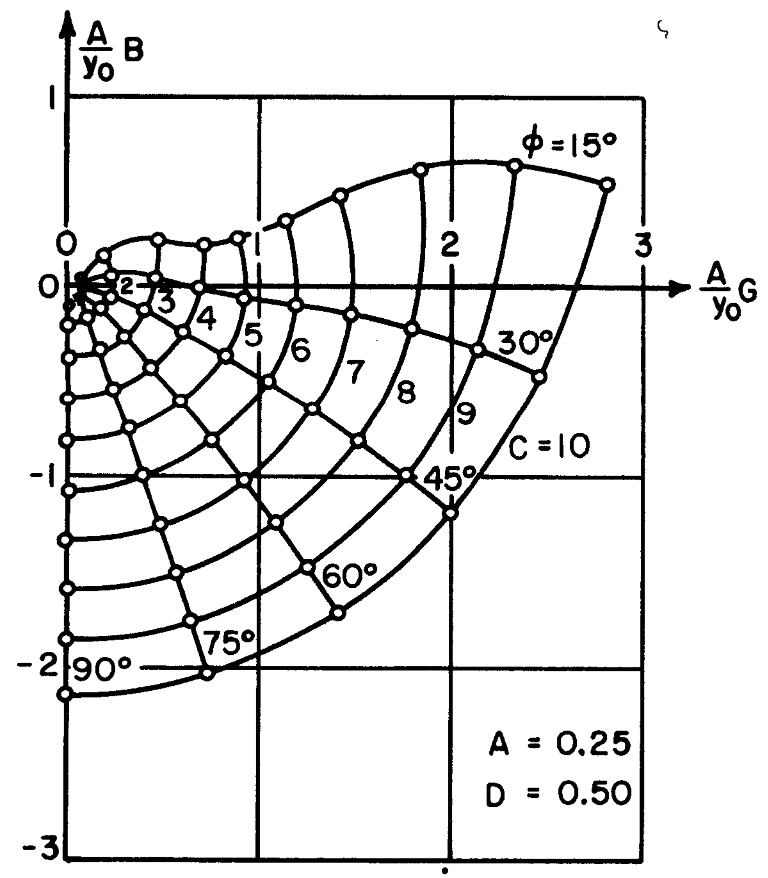

Fig. 20--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

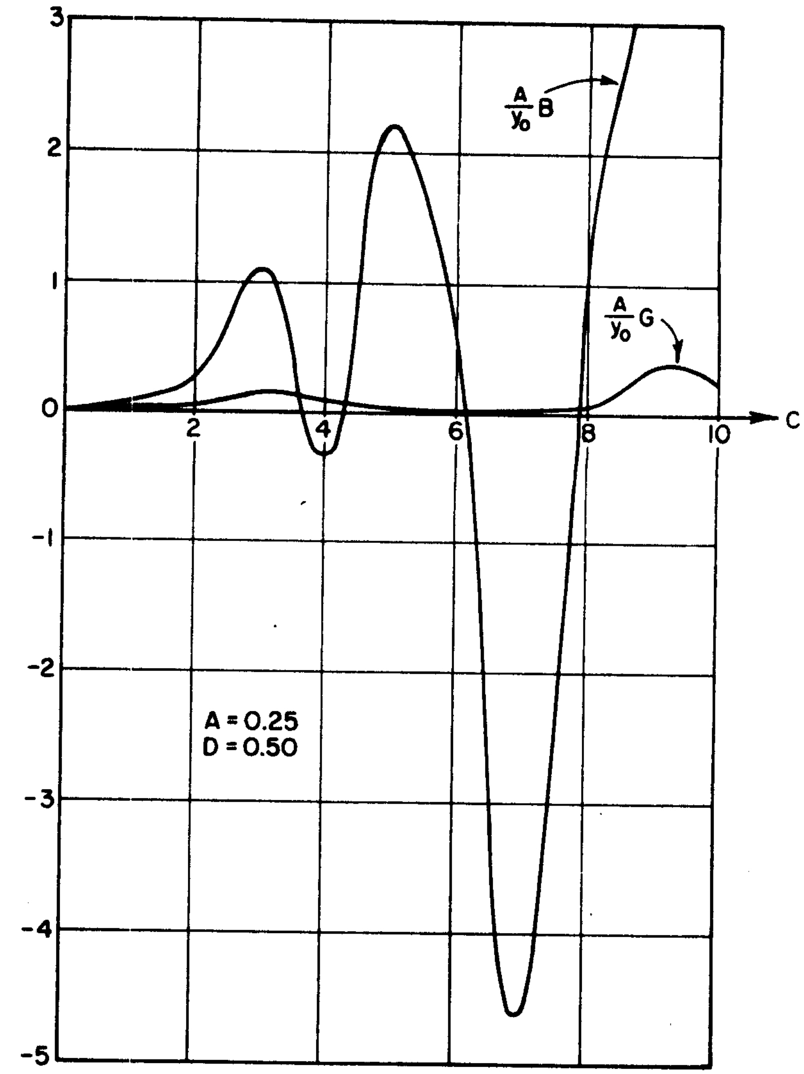

Fig. 21--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

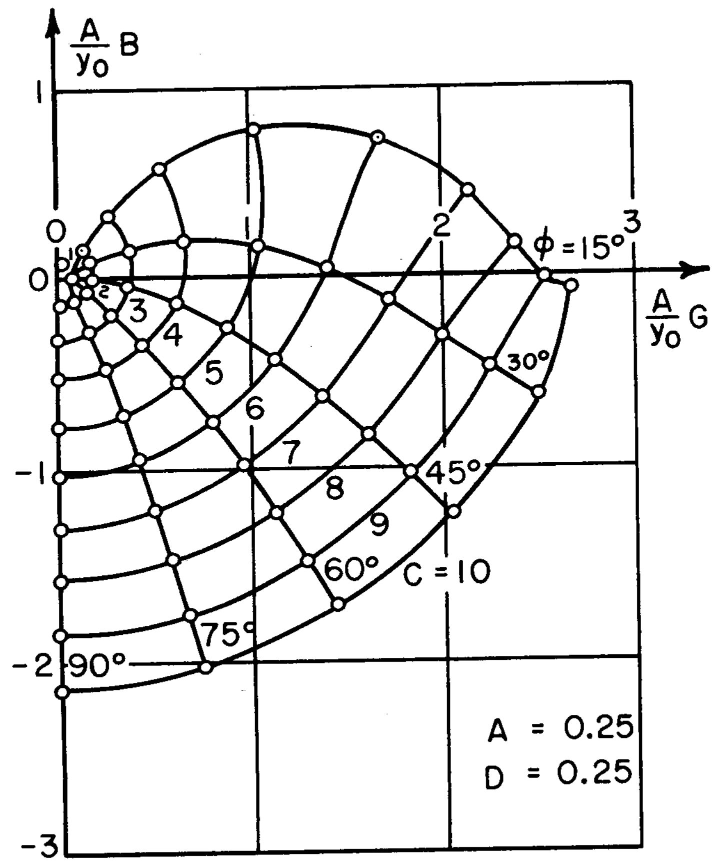

Fig. 22--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

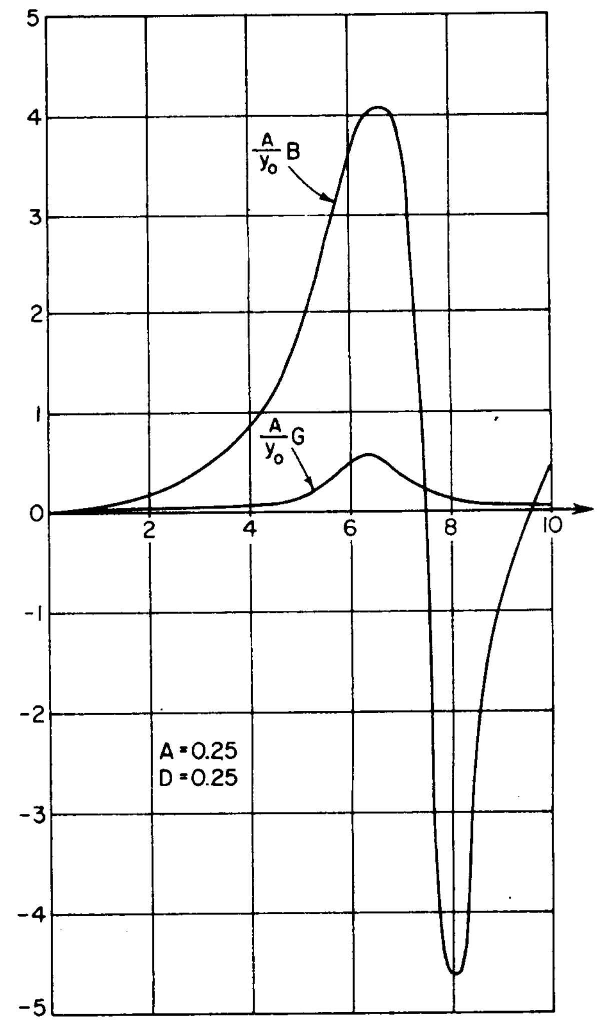

Fig. 23--The normalized admittance $\frac{A}{Y_o}Y$ for a finite slab

CHAPTER IV
THE RECTANGULAR APERTURE

In this section the admittance of a rectangular aperture radiating into a lossy medium will be found. As in Chapter I, Part A treats the case of an infinite lossy medium and Part B the case of a lossy slab.

A. The Infinite Lossy Medium

Consider a rectangular waveguide which radiates through an opening in an infinite ground screen, as shown in Fig. 20. The half-space $z > 0$ is assumed to be homogeneous and isotropic, with a complex propagation constant k_1 as given in Eq. (35).

The waveguide aperture has dimensions (a, b), as shown in Fig. 24. The electric field in the aperture is assumed to have the form of the TE_{10} waveguide mode, with the electric field in the x-direction

$$(158) \qquad E_x(x, y, o) = \overline{e}_o(x, y) = \begin{cases} \sqrt{\dfrac{2}{ab}} \cos \dfrac{\pi y}{b} & : (x, y) \in \text{Aperture} \\ 0 & : (x, y) \notin \text{Aperture.} \end{cases}$$

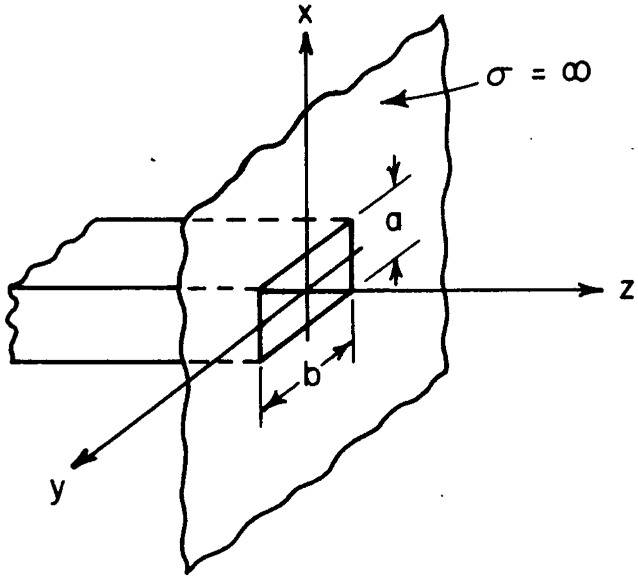

Fig. 24--Rectangular aperture in a ground plane

The normalizing constant $\sqrt{\frac{2}{ab}}$ is included so that the normalization relations

$$(159)\quad \int_{x=-a/2}^{a/2}\int_{y=-b/2}^{b/2} |\bar{e}_t|^2\,dx\,dy = \int_{x=-a/2}^{a/2}\int_{y=-b/2}^{b/2} |\bar{h}_t|^2\,dx\,dy = 1$$

are satisfied, as discussed in Chapter I.

With the aperture field as given in (158), the field is everywhere TE to the y-axis.[42] Hence the field may be represented by an electric vector potential

$$(160)\qquad \bar{F} = \hat{y}\,\psi$$

---

[42] For a proof of this, see Reference 37.

where ψ satisfies the wave equation

(161) $$\nabla^2 \psi + k_1^2 \psi = 0$$

with the appropriate boundary conditions. The electric and magnetic fields are given by

(162) $$\overline{E} = -\nabla \times \overline{F}$$

(163) $$\overline{H} = \frac{1}{j\omega\mu_o}\left[\nabla(\nabla \cdot \overline{F}) + k_1^2 \overline{F}\right].$$

In particular,

(164) $$E_x = \frac{\partial\psi}{\partial y}$$

(165) $$H_y = \frac{1}{j\omega\mu_o}\left[\frac{\partial^2\psi}{\partial y^2} + k_1^2 \psi\right].$$

A solution for ψ is given by:

(166) $$\psi = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x, k_y)\, e^{-jk_z z}\, e^{-jk_x x}\, e^{-jk_y y}\, dk_x\, dk_y$$

with

(167) $$k_z = \sqrt{k_1^2 - k_x^2 - k_y^2}$$

where the square root is chosen so that

$$\text{(168)} \qquad \operatorname{Re}(k_z) \geq 0$$

$$\text{(169)} \qquad \operatorname{Im}(k_z) \leq 0$$

corresponding to propagation in the +z-direction. Then from (164),

$$\text{(170)} \qquad E_x(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -jk_z f(k_x, k_y) e^{-jk_z z} e^{-jk_x x} e^{-jk_y y} \cdot dk_x \, dk_y \, .$$

The inverse transform, evaluated at z = 0, gives

$$\text{(171)} \qquad -jk_z f(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x(x, y, o) e^{+jk_x x} e^{+jk_y y} \, dx \, dy \, .$$

Substituting for $E_x(x, y, o)$ from (158) results in

$$\text{(172)} \qquad -jk_z f(k_x, k_y) = \sqrt{\frac{2}{ab}} \int_{x=-a/2}^{a/2} \int_{y=-b/2}^{b/2} \cos \frac{\pi y}{b} \, e^{+jk_x x} e^{+jk_y y} dx \, dy$$

$$= \sqrt{\frac{2}{ab}} \left[\frac{2}{k_x} \sin\left(\frac{k_x a}{2}\right)\right] \left[\frac{2\pi b \cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2}\right] .$$

Hence,

$$\text{(173)} \qquad f(k_x, k_y) = \frac{4\pi j}{k_x k_z} \sqrt{\frac{2b}{a}} \, \frac{\sin\left(\frac{k_x a}{2}\right) \cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2}$$

and ψ in (166) is then

$$(174) \quad \psi = \frac{1}{(2\pi)^2}\sqrt{\frac{2b}{a}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{4\pi y}{k_x k_z}\;\frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2}$$

$$\cdot\; e^{-jk_z z} e^{-jk_x x}\, e^{-jk_y y}\, dk_x\, dk_y\, .$$

Then from (164) and (165), E_x and H_y are found to be:

$$(175) \quad E_x(x, y, z) = \frac{1}{(2\pi)^2}\sqrt{\frac{2b}{a}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{4\pi}{k_x}\,\frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2}$$

$$\cdot\; e^{-jk_z z}\, e^{-jk_x x}\, e^{-jk_y y}\, dk_x\, dk_y$$

$$(176) \quad H_y(x, y, z) = \frac{1}{(2\pi)^2}\sqrt{\frac{2b}{a}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{4\pi\,(k_1^2 - k_y^2)}{k_x k_z}\;\frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2}$$

$$\cdot\; e^{-jk_z z}\, e^{-jk_x x}\, e^{-jk_y y}\, dk_x\, dk_y\, .$$

The aperture admittance may now be calculated from Eq. (12), which for this case is

$$(177) \quad Y = \frac{\displaystyle\int_{x=-a/2}^{a/2}\int_{y=-b/2}^{b/2}\overline{E}(x, y, o) \times \overline{\Gamma}(x, y, o) \cdot \hat{z}\, dx\, dy}{\left[\displaystyle\int_{x=-a/2}^{a/2}\int_{y=-b/2}^{b/2}\overline{E}(x, y, o) \cdot \overline{e}_o(x, y)\, dx\, dy\right]^2}\, .$$

As with the infinite slot, since $\overline{e}_o(x, y)$ is used for $\overline{E}(x, y, o)$, it follows from the orthogonality of the vector mode functions that

$$(178) \quad \int_{x=-a/2}^{a/2} \int_{y=-b/2}^{b/2} \overline{E}(x, y, o) \times \overline{\Gamma}(x, y, o) \cdot \hat{z}\, dx\, dy$$

$$= \int_{x=-a/2}^{a/2} \int_{y=-b/2}^{b/2} \overline{E}(x, y, o) \times \overline{H}(x, y, o) \cdot \hat{z}\, dx\, dy \; .$$

Also the integral in the denominator of (177) is unity. Hence, substituting (175) and (176) in (178) and making use of Parseval's theorem[43] (the limits of integration may be extended to infinity because

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[43] For the Fourier transform pair

$$g(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x, k_y) e^{-jk_x x} e^{-jk_y y}\, dk_x\, dk_y$$

$$G(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{+jk_x x}\, e^{+jk_y y}\, dx\, dy$$

Parseval's theorem is:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x, y) g_2^*(x, y) dx\, dy = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_1(k_x, k_y) G_2^*(k_x, k_y) \cdot dk_x\, dk_y$$

and one form of the convolution theorem is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x, y) g_2(-x, -y) dx\, dy = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_1(k_x, k_y) G_2(k_x, k_y) \cdot dk_x\, dk_y$$

$\overline{E}(x, y, o)$ is zero outside the aperture) yields for the admittance:

$$(179) \quad Y = \frac{2b}{(2\pi)^2 a\omega\mu_o} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(4\pi)^2 (k_1^2 - k_y^2)}{\pi^2 k_x^2 b_z} \frac{\sin^2\left(\frac{k_x a}{2}\right) \cos^2\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)^2} \cdot dk_x \, dk_y .$$

Next, the terms of the integrand may be recombined as follows. Let

$$(180) \quad F_1(k_x, k_y) = \frac{8b(k_1^2 - k_y^2)}{a\omega\mu_o k_x^2} \frac{\sin^2 \frac{k_x a}{2} \cos^2 \frac{k_y b}{2}}{(\pi^2 - k_y^2 b^2)^2}$$

$$(181) \quad F_2(k_x, k_y) = \frac{4\pi^2}{k_z} = \frac{4\pi^2}{\sqrt{k_1^2 - k_x^2 - k_y^2}} .$$

Then using the form of the convolution theorem given in the footnote on p. 62,

$$(182) \quad Y = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(k_x, k_y) F_2(k_x, k_y) \, dk_x \, dk_y$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x, y) f_2(-x, -y) dx \, dy$$

where $f_1(x, y)$, $f_2(x, y)$ are the transforms of $F_1(k_x, k_y)$, $F_2(k_x, k_y)$, and are found below.

Consider $f_1(x, y)$ first:

$$(183)\qquad f_1(x, y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(k_x, k_y)e^{-jk_x x}e^{-jk_y y}\,dk_x\,dk_y$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{8b(k_1^2 - k_y^2)}{a\omega\mu_o k_x^2}\,\frac{\sin^2\left(\frac{k_x a}{2}\right)\cos^2\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)^2}$$

$$\cdot\, e^{-jk_x x}e^{-jk_y y}\,dk_x\,dk_y$$

$$(184)\qquad = \frac{2b}{a\omega\mu_o\pi^2}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{k_x a}{2}\right)}{k_x^2}e^{-jk_x x}\int_{-\infty}^{\infty}(k_1^2 - k_y^2)\,\frac{\cos^2\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)^2}$$

$$\cdot\, e^{-jk_y y}\,dk_y\,.$$

These integrals are easily done, and the result is:

$$(185)\qquad f_1(x, y) = \frac{2b}{a\omega\mu_o\pi^2}\,g(x)h(y)$$

where

$$(186)\qquad g(x) = \begin{cases} \frac{\pi}{2}(a - |x|) : & |x| \le a \\ 0 \qquad : & |x| > a \end{cases}$$

$$(187)\qquad h(y) = \begin{cases} D_1(b - |y|)\cos\frac{\pi y}{b} + D_2\sin\frac{\pi|y|}{b} & : \ |y| \le b \\ 0 & : \ |y| > b \end{cases}$$

and

$$(188) \quad D_1 = \frac{1}{b^2}\left[\frac{k_1^2}{4\pi} - \frac{\pi}{4b^2}\right]$$

$$(189) \quad D_2 = \frac{1}{b^2}\left[\frac{bk_1^2}{4\pi^2} + \frac{1}{4b}\right] = \frac{1}{\pi b}\left[\frac{k_1^2}{4\pi} + \frac{\pi}{4b^2}\right].$$

Next, for $f_2(x, y)$,

$$(190) \quad f_2(x, y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(k_x, k_y) e^{-jk_x x} e^{-jk_y y}\, dk_x\, dk_y$$

$$(191) \quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk_x x} e^{-jk_y y}}{\sqrt{k_1^2 - k_y^2 - k_x^2}}\, dk_x\, dk_y \; .$$

Doing the integration on k_x gives[44]

$$(192) \quad \int_{-\infty}^{\infty} \frac{e^{-jk_x x}\, dk_x}{\sqrt{k_1^2 - k_y^2 - k_x^2}} = +\pi H_o^{(2)}\left(|x|\sqrt{k_1^2 - k_y^2}\right) \; .$$

The integration on k_y then yields

$$(193) \quad f_2(x, y) = \int_{-\infty}^{\infty} \left[\pi H_o^{(2)}\left(|x|\sqrt{k_1^2 - k_y^2}\right)\right] e^{-jk_y y}\, dk_y$$

$$(194) \quad = 2\pi j \frac{e^{-jk_1\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}}$$

---

[44] This integral is derived in Reference 37.

The integral in (193) is known as Weyrich's integral.[45] Thus, (182) gives

$$(195)\qquad Y = \int_{y=-b}^{b}\int_{x=-a}^{a} \frac{4bj}{a\pi\mu_o\pi}\, g(x)\,h(y)\, \frac{e^{-jk_1\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\,dx\,dy$$

and since $g(x) = g(-x)$ and $h(y) = h(-y)$, Y may be written:

$$(196)\qquad Y = \frac{16bj}{a\omega\mu_o\pi}\int_{y=0}^{b}\int_{x=0}^{a} g(x)h(y)\,\frac{e^{-jk_1\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\,dx\,dy\,.$$

Finally, substituting for $g(x)$, $h(y)$,

$$(197)\qquad Y = \frac{8bj}{a\omega\mu_o}\int_{y=0}^{b}\int_{x=0}^{a}(a-x)\left[D_1(b-y)\cos\frac{\pi y}{b} + D_2\sin\frac{\pi y}{b}\right] \cdot \frac{e^{-jk_1\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\,dx\,dy\,.$$

Before evaluating (197), it is convenient to normalize with respect to the free-space constants. As before, let k_o be the free-space propagation constant,

$$(198)\qquad k_o = \omega\sqrt{\mu_o\epsilon_o} = \frac{2\pi}{\lambda_o}$$

[45] W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Functions of Mathematical Physics," Chelsea Publ. Co., New York, 1954; p. 34.

(λ_o is the free-space wavelength), and let y_o be the free-space characteristic admittance,

$$(199) \qquad y_o = \sqrt{\frac{\epsilon_o}{\mu_o}} \quad .$$

Then (197) may be written

$$(200) \qquad Y_n = \frac{Y}{y_o} = 8\,\frac{B}{A}\,j \int_{\eta=0}^{A} \int_{\xi=0}^{B} (A-\eta)\left[C_1\,(B-\xi)\cos\frac{\pi\xi}{B}\ \sin\frac{\pi\xi}{B}\right] \cdot \frac{e^{-j\left(\frac{k_1}{k_o}\right)\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}}\, d\eta\, d\xi$$

where

$$(201) \qquad A = k_o a$$

$$(202) \qquad B = k_o b$$

$$(203) \qquad C_1 = \frac{D_1}{k_o^4} = \frac{1}{4\pi B^2}\left[\left(\frac{k_1}{k_o}\right)^2 - \left(\frac{\pi}{B}\right)^2\right]$$

$$(204) \qquad C_2 = \frac{D_2}{k_o^3} = \frac{1}{4\pi^2 B}\left[\left(\frac{k_1}{k_o}\right)^2 + \left(\frac{\pi}{B}\right)^2\right]$$

and Y_n is the normalized aperture admittance.[46]

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[46] Note that the normalization is not the same as for the infinite slot.

B. Numerical Results and Interpretation

Equation (200) has been evaluated in the University's Numerical Computation Laboratory with the IBM 1620 Digital Computer for three sizes of apertures:

(a) $A = \frac{\pi}{2}, \quad B = \pi\left(\frac{\lambda_o}{4} \text{ by } \frac{\lambda_o}{2}\right)$

(b) $A = \frac{3\pi}{4}, \quad B = \frac{3\pi}{2}\left(\frac{3\lambda_o}{8} \text{ by } \frac{3\lambda_o}{4}\right)$

(c) $A = \pi, \quad B = 2\pi\left(\frac{\lambda_o}{2} \text{ by } \lambda_o\right)$.

The computation was done by means of Simpson's rule, after making the change of variables

(205) $$\eta = R \cos \theta$$

(206) $$\xi = R \sin \theta .$$

With this substitution, (200) becomes

(207) $$Y_n = 8 \frac{B}{A} j \int_{\theta=0}^{\theta_o} \int_{R=0}^{\frac{A}{\cos \theta}} (A - R\cos\theta)\left[C_1(B - R\sin\theta)\cos\frac{\pi}{B}(R\sin\theta) + C_2 \sin\frac{\pi}{B}(R\sin\theta)\right] e^{-j\frac{k_1}{k_o}R}\, dR\, d\theta$$

(207) cont.
$$+8\,\frac{B}{A}\,j\int_{\theta=\theta_o}^{\frac{\pi}{2}}\int_{R=0}^{\frac{B}{\sin\theta}}(A-R\cos\theta)\left[C_1\,(B-R\sin\theta)\cos\frac{\pi}{B}(R\ \sin\theta)\right.$$
$$\left.+C_2\sin\frac{\pi}{B}(R\sin\theta)\right]e^{-j\frac{k_1}{k_o}R}\,dR\,d\theta$$

where

(208) $\tan\theta_o = \dfrac{B}{A}$.

This change of variables removes the singularity at $\eta = \xi = 0$ in the integrand of (200), which is troublesome for computer evaluation.

The double integral is evaluated as an iterated integral, the integration of R being done first. Simpson's Rule is used throughout. First, with θ held constant at each of the values 0, $0.1\left(\frac{\pi}{2}\right)$, $0.2\left(\frac{\pi}{2}\right)$, ..., $\left(\frac{\pi}{2}\right)$, the R-integral is computed by breaking the range of R into ten subintervals, evaluating the integrand at the end-points of the subintervals, and summing according to Simpson's Rule. These values, which form the integrand for the θ-integral, are then summed again by Simpson's Rule to evaluate the θ-integral.

The results of this calculation are shown in Figs. 25 through 29. The admittance Y_n is plotted in terms of normalized conductance G_n and normalized susceptance B_n:

(209) $$Y_n = G_n + j\,B_n\ .$$

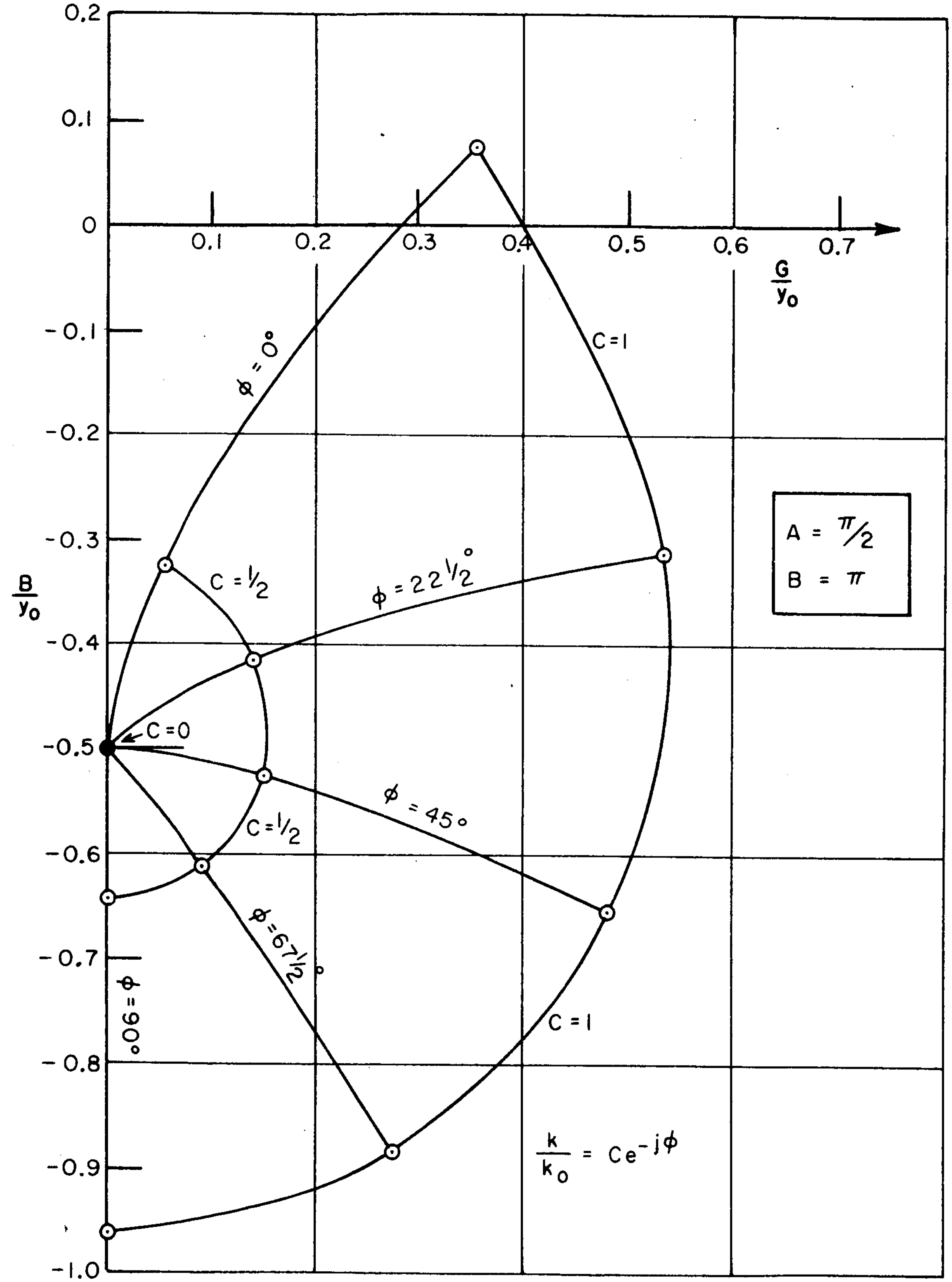

Fig. 25--The normalized admittance $\frac{Y}{Y_o}$ for an infinite medium

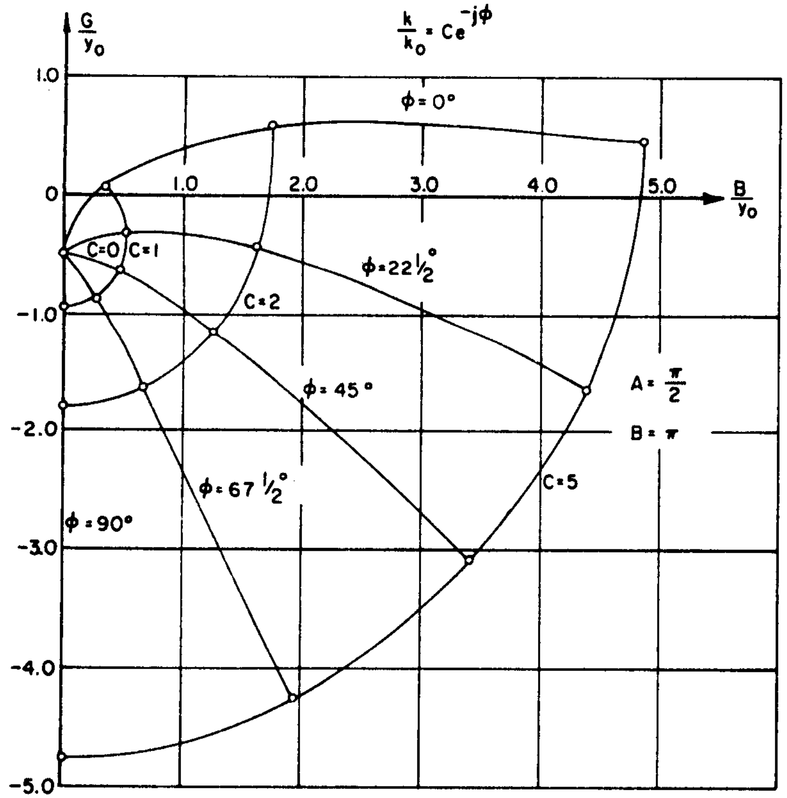

Fig. 26--The normalized admittance $\frac{Y}{Y_o}$ for an infinite medium

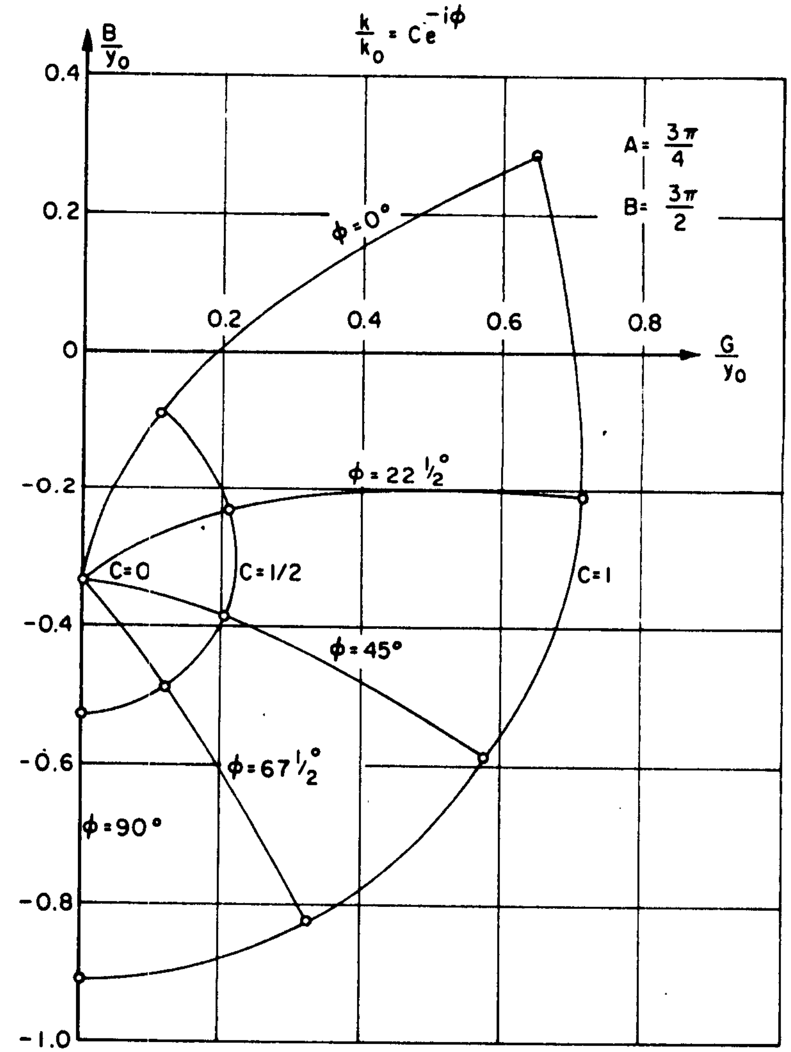

Fig. 27--The normalized admittance $\frac{Y}{Y_o}$ for an infinite medium

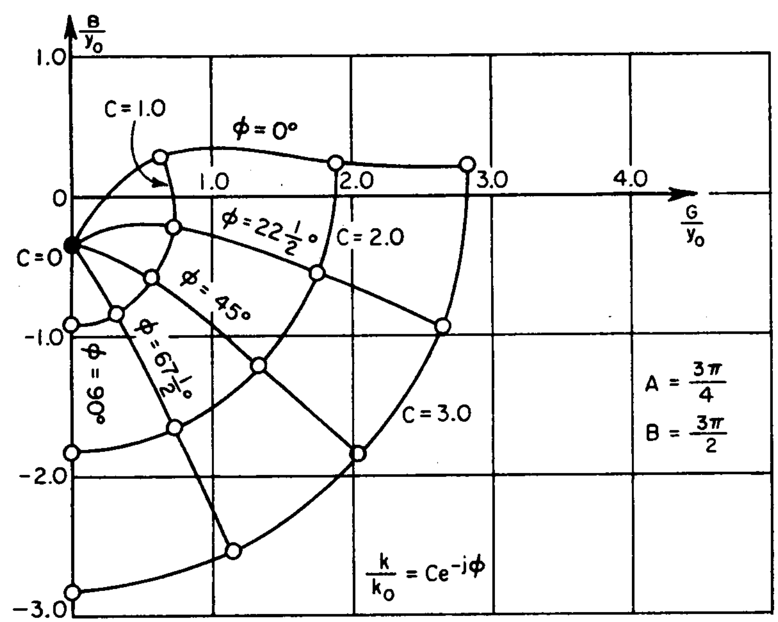

Fig. 28--The normalized admittance $\frac{Y}{Y_o}$ for an infinite medium

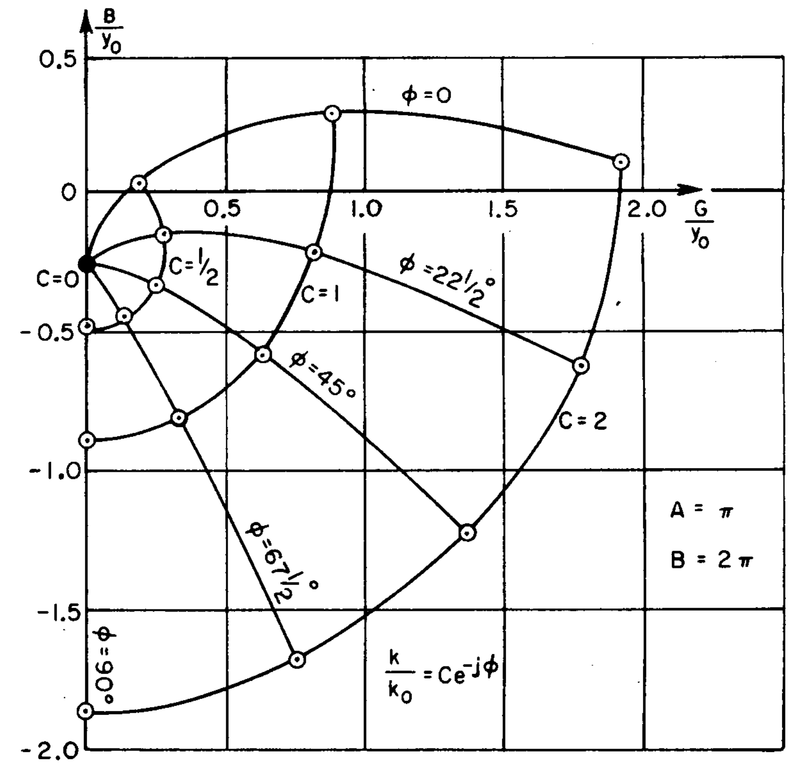

Fig. 29--The normalized admittance $\frac{Y}{Y_o}$ for an infinite medium

With C and ϕ defined by

(210) $$\frac{k_1}{k_o} = C e^{-j\phi} .$$

the results are given for various values of C and ϕ.

Figures 25 and 26 show Y_n for the case $A = \frac{\pi}{2}$, $B = \pi$. Figure 25 gives the results for $0 \leq C \leq 1$ and $0 \leq \phi \leq 90°$, and Fig. 26 for $0 \leq C \leq 5$ and $0 \leq \phi \leq 90°$. Figures 27 and 28 show Y_n for the case $A = \frac{3\pi}{4}$, $B = \frac{3\pi}{2}$. In Fig. 27, the limits are $0 \leq C \leq 1$ and $0 \leq \phi \leq 90°$ and in Fig. 28, $0 \leq C \leq 3$. Figure 29 shows Y_n for $A = \pi$, $B = 2\pi$ and $0 \leq C \leq 2$, $0 \leq \phi \leq 90°$.

Finally, as a check on the numerical results, the integral for Y_n may be evaluated approximately for the case where k_1 has a large (complex) value. In (200),

(211) $$Y_n = \frac{Y}{Y_o} = 8\,\frac{B}{A}\,j \int_{\eta=0}^{A} \int_{\xi=0}^{B} (A-\eta)\left[C_1 (B-\xi)\cos\frac{\pi\xi}{B} + C_2 \sin\frac{\pi\xi}{B}\right] \cdot \frac{e^{-j\frac{k_1}{k_o}\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}}\, d\eta\, d\xi .$$

The change of variables in Eqs. (205), (206) gives the substitution

(212) $$\frac{e^{-j\frac{k_1}{k_o}\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}}\, d\eta\, d\xi = e^{-j\left(\frac{k_1}{k_o}\right) R}\, dR\, d\theta .$$

If $\frac{k_1}{k_o}$ has a large (negative) imaginary part, the only contribution to the integral in (211) will occur in the vicinity of $R = 0$. In this region the other terms in the integrand may be approximated by

$$(A - \eta) \cong A \tag{213}$$

$$C_1 (B-\xi) \cos \frac{\pi\xi}{B} \cong C_1 B \tag{214}$$

$$C_2 \sin \frac{\pi\xi}{B} \cong 0 \ . \tag{215}$$

Also the range of integration on R may be extended to infinity with little change in the value of the integral. With these simplifications (211) becomes

$$\frac{Y}{Y_o} = 8 \ \frac{B}{A} \ j \int_{R=0}^{\infty} \int_{\theta=0}^{\frac{\pi}{2}} A \, C_1 \, B \, e^{-j\left(\frac{k_1}{k_o}\right)R} \ dR \ d\theta \tag{216}$$

$$= 4\pi C_1 B^2 \left(\frac{k_1}{k_o}\right) . \tag{217}$$

From (203), for large $\frac{k_1}{k_o}$, C_1 becomes

$$C_1 \cong \frac{1}{4\pi B^2 \left(\frac{k_1}{k_o}\right)^2} \tag{218}$$

so (217) yields

$$\frac{Y}{Y_o} \cong \frac{k_1}{k_o} \tag{219}$$

a surprisingly simple result. This behavior for large $\frac{k_1}{k_o}$ is clearly indicated in Figs. 25 through 29.

For small $\frac{k_1}{k_o}$ it is difficult to find a simple approximation for Y_n from Eq. (200). However, for the case where $\frac{k_1}{k_o}$ is purely imaginary, it is easy to see that Eq. (200) gives a purely imaginary admittance, because C_1 and C_2 are real and the integrand has a real value.

The reason for this can be appreciated by examining Eq. (176) for the magnetic field. In the aperture, (176) gives:

$$H_y(x, y, o) = \frac{1}{(2\pi)^2} \frac{1}{\omega\mu_o} \sqrt{\frac{2b}{a}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4\pi(k_1^2 - k_y^2)}{k_x k_z} \frac{\sin\left(\frac{ka}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2} \cdot e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y . \tag{220}$$

This may be written

$$H_y(x, y, o) = \frac{1}{(2\pi)^2} \frac{1}{\omega\mu_o} \sqrt{\frac{2b}{a}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_1(k_x, k_y) G_2(k_x, k_y) \cdot e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y \tag{221}$$

where

$$(222) \quad G_1(k_x, k_y) = \frac{1}{k_z} = \frac{1}{\sqrt{k_1 - k_x - k_y}}$$

$$(223) \quad G_2(k_x, k_y) = \frac{4\pi(k_1^2 - k_y^2)}{k_x} \quad \frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2} \quad .$$

Then by making use of the convolution theorem

$$(224) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(\eta, \xi) g_2(x-\eta, y-\xi) d\eta \, d\xi$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(k_x, k_y) G_2(k_x, k_y) e^{-jk_x x} e^{-jk_y y} dk_x \, dk_y$$

where

$$(225) \quad g_1(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(k_x, k_y) e^{-jk_x x} e^{-jk_y y} dk_x \, dk_y$$

$$(226) \quad g_2(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_2(k_x, k_y) e^{-jk_x x} e^{-jk_y y} dk_x \, dk_y,$$

and the transform pairs given by Eq. (175) and Eq. (190) through (194), H_y may be written

$$(227) \quad H_y(x, y, o) = \frac{1}{\omega\mu_o} \frac{j}{2\pi} \sqrt{\frac{2}{ab}} \left(k_1^2 + \frac{\partial^2}{\partial y^2} \right) \int_{\eta=-a/2}^{a/2} \int_{\xi=-b/2}^{b/2}$$

$$\cdot \cos \frac{\pi}{b} (y-\xi) \frac{e^{-jk_1\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}} \, d\eta \, d\xi \, .$$

Now for the case where $Re(k_1) = 0$, the integrand in (227) is real, k_1^2 is real, and hence H_y is purely imaginary. This means that the electric and magnetic fields in the aperture are in time quadrature. The integral in the numerator of Eq. (177) is therefore imaginary.

This situation is similar to the case of large waveguide terminated by a small cutoff waveguide, as illustrated in Fig. 30.

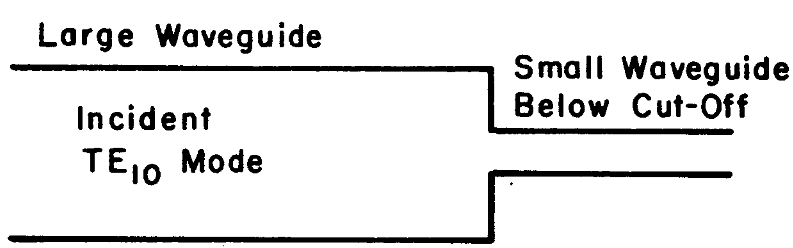

Fig. 30--Waveguide analogy

In the small waveguide the electric and magnetic fields are in phase quadrature and the effective termination of the large guide is a pure susceptance.

It is interesting to note that for $k_1 = 0$, the magnetic field is quasi-static. For a fixed aperture size and fixed frequency, the condition $k_1 = 0$ corresponds to $\epsilon_1 = 0$, $\sigma_1 = 0$ in Eq. (35), which leads to a Laplace's equation for the magnetic field. (The case $k_1 = 0$ can also be interpreted as the zero-frequency limit; but since the curves in Fig. 25 through Fig. 29 are plotted for constant $A = k_o a$ and $B = k_o b$, the physical aperture size must be considered as varying inversely with frequency in this case.)

C. The Lossy Slab

Next the aperture will be assumed to radiate through a lossy slab of thickness d into free-space. The field in the aperture is the same as in Part A, as given in Eq. (158).

Unlike the previous cases treated, the electromagnetic fields for this problem are not TE to the y-axis. An attempt to construct a solution with $\overline{F}$ as defined in Eq. (125) will not work, because the solution will not satisfy all boundary conditions. Hence the vector potential must have two components. One possible choice for $\overline{F}$ is

$$(228) \qquad \overline{F} = \phi\hat{x} + \psi\hat{y}$$

where ϕ and ψ are both solutions to the wave equation

$$(229) \qquad (\nabla^2 + k_{1,o}^2)\begin{Bmatrix} \psi \\ \phi \end{Bmatrix} = 0 \; .$$

As in Part I-B above, the slab will be called Region 1 and the free-space Region 0. In region 1, solution for ψ and ϕ may be constructed in the form

$$(230) \qquad \psi_1(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [I_\psi e^{-jk_{z1}z} + R_\psi e^{+jk_{z1}z}] \cdot e^{-jk_x x \; -jk_y y} \, dk_x \, dk_y$$

$$(231) \qquad \phi_1(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [I_\phi e^{-jk_{z1}z} + R_\phi e^{+jk_{z1}z}] \cdot e^{-jk_x x \; -jk_y y} \, dk_x \, dk_y$$

and in Region 0,

$$(232) \qquad \psi_o(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_\psi e^{-jk_{zo}z} \, e^{-jk_x x} \, e^{-jk_y y} \, dk_x \, dk_y$$

$$(233) \quad \phi_o(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\phi e^{-jk_{zo}z} \; e^{-jk_x x} \; e^{-jk_y y} \, dk_x \, dk_y$$

where the propagation constants

$$(234) \quad k_{z1} = \sqrt{k_1^2 - k_x^2 - k_y^2}$$

$$(235) \quad k_{zo} = \sqrt{k_o^2 - k_x^2 - k_y^2}$$

are chosen so that

$$(236) \quad \mathrm{Re}(k_{z1}), \quad \mathrm{Re}(k_{zo}) \geq 0$$

$$(237) \quad \mathrm{Im}(k_{z1}), \quad \mathrm{Im}(k_{zo}) \leq 0 \; .$$

The fields are found from the relations

$$(238) \quad \overline{E} = -\nabla \times \overline{F}$$

$$(239) \quad \overline{H} = \frac{1}{j\omega\mu_o} \left[k_{1,o}^2 \, \overline{F} + \nabla (\nabla \cdot \overline{F}) \right]$$

which give in Region 1,

$$(240) \quad E_{x1}(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[-jk_{z1} I_\psi e^{-jk_{z1}z} + jk_{z1} R_\psi e^{+jk_{z1}z} \right] \cdot e^{-jk_x x} e^{-jk_y y} \, dk_x \, dk_y$$

$$(241)\quad E_{y1}(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[+jk_{z1}I_\phi e^{-jk_{z1}z} - jk_{z1}R_\phi e^{+jk_{z1}z}\right] \cdot e^{-jk_x x \;-jk_y y} dk_x\, dk_y$$

$$(242)\quad H_{x1}(x,y,z) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{k_1^2-k_x^2}{j\omega\mu_o}\left[I_\phi e^{-jk_{z1}z} + R_\phi e^{+jk_{z1}}\right] - \frac{k_x k_y}{j\omega\mu_o}\left[I_\psi e^{-jk_{z1}z} + R_\psi e^{+jk_{z1}z}\right]\right\} \cdot e^{-jk_x x}\, e^{-jk_y y}\, dk_x\, dk_y$$

$$(243)\quad H_{y1}(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{k_1^2-k_y^2}{j\omega\mu_o}\left[I_\psi e^{-jk_{z1}z} + R_\psi e^{+jk_{z1}z}\right] - \frac{k_x k_y}{j\omega\mu_o}\left[I_\phi e^{-jk_{z1}z} + R_\phi e^{+jk_{z1}z}\right]\right\} \cdot e^{-jk_x x}\, e^{-jk_y y}\, dk_x\, dk_y$$

and in region 0,

$$(244)\quad E_{xo}(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -jk_{zo}\, T_\psi e^{-jk_{zo}z} e^{-jk_x x} e^{-jk_y y} dk_x\, dk_y$$

$$(245)\qquad E_{yo}(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} +jk_{zo}\, T_\phi e^{-jk_{zo}z} e^{-jk_x x} e^{-jk_y y} \cdot dk_x\, dk_y$$

$$(246)\qquad H_{xo}(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{k_o^2-k_x^2}{j\omega\mu_o} T_\phi - \frac{k_x k_y}{j\omega\mu_o} T_\psi\right] \cdot e^{-jk_{zo}z} e^{-jk_x x} e^{-jk_y y}\, dk_x\, dk_y$$

$$(247)\qquad H_{yo}(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{k_o^2-k_y^2}{j\omega\mu_o} T_\psi - \frac{k_x k_y}{j\omega\mu_o} T_\phi\right] \cdot e^{-jk_{zo}z \; -jk_x x \; -jk_y y}\, dk_x\, dk_y .$$

Taking the inverse transform of (240) and (241) at $z = 0$ gives

$$(248)\qquad jk_{z1}\left[-I_\psi + R_\psi\right] = 4\pi\sqrt{\frac{2b}{a}}\; \frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{k_x(\pi^2 - k_y^2 b^2)}$$

$$(249)\qquad jk_{z1}\left[I_\phi - R_\phi\right] = 0 .$$

Applying boundary conditions at $z = d$ leads to the four equations

$$(250)\qquad -jk_{z1} I_\psi e^{-jk_{z1}d} + jk_{z1} R_\psi e^{+jk_{z1}d} = -jk_{zo} T_\psi e^{-jk_{zo}d}$$

$$(251)\qquad jk_{z1} I_\phi e^{-jk_{z1}d} - jk_{z1} R_\phi e^{+jk_{z1}d} = +jk_{zo} T_\phi e^{-jk_{zo}d}$$

$$(252)\qquad (k_1^2-k_x^2)\,[I_\phi e^{-jk_{z1}d}+R_\phi e^{+jk_{z1}d}]-k_xk_y[I_\psi e^{-jk_{z1}d}+R_\psi e^{+jk_{z1}d}]$$

$$= (k_o^2-k_x^2)\,T_\phi e^{-jk_{zo}d}-k_xk_y\,T_\psi e^{-jk_{zo}d}$$

$$(253)\qquad (k_1^2-k_y^2)\,[I_\psi e^{-jk_{z1}d}+R_\psi e^{+jk_{z1}d}]-k_xk_y[I_\phi e^{-jk_{z1}d}+R_\phi e^{+jk_{z1}d}]$$

$$= (k_o^2-k_y^2)\,T_\psi e^{-jk_{zo}d}-k_xk_y\,T_\phi e^{-jk_{zo}d}.$$

The solution of Eqs. (248) through (253) is tedious but straightforward. In determinant form, I_ψ and I_ϕ are found to be given by

$$(254)\qquad I_\psi = \frac{\begin{vmatrix} A_{11} & B_1 \\ A_{12} & B_2 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix}}$$

$$(255)\qquad I_\phi = \frac{\begin{vmatrix} B_1 & A_{12} \\ B_2 & A_{22} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix}}$$

where

$$(256)\qquad B_1 = -k_xk_y\,e^{+jk_{z1}d}\left(\frac{k_{z1}+k_{zo}}{2k_{zo}}\right)jf$$

$$(257)\qquad B_2 = \frac{1}{2}\,e^{+jk_{z1}d}\left\{(k_o^2-k_y^2)+(k_1^2-k_y^2)\,\frac{k_{zo}}{k_{z1}}\right\}jf$$

$$(258)\qquad A_{11} = (k_1^2 - k_x^2)k_{zo}\cos k_{z1}d + j(k_o^2 - k_x^2)k_{z1}\sin k_{z1}d$$

$$(259)\qquad A_{12} = -k_x k_y(k_{zo}\cos k_{z1}d + jk_{z1}\sin k_{z1}d)$$

$$(260)\qquad A_{22} = (k_1^2 - k_y^2)k_{zo}\cos k_{z1}d + j(k_o^2 - k_y^2)k_{z1}\sin k_{z1}d$$

and where f is the quantity

$$(261)\qquad f = 4\pi\sqrt{\frac{2b}{a}}\;\frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{k_x(\pi^2 - k_y^2 b^2)} \quad .$$

Also from (248) and (249), R_ψ and R_ϕ are given by

$$(262)\qquad R_\psi = \frac{f}{jk_{z1}} + I_\psi$$

$$(263)\qquad R_\phi = I_\phi \quad .$$

Equation (177) of part A will be used to find the admittance. Using Parseval's theorem and Eqs. (240) and (243) for the fields gives

$$(264)\qquad Y = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{jk_{z1}[-I_\psi + R_\psi]\right\}^*\left\{\frac{1}{j\omega\mu_o}[k_1^2 - k_y^2)(I_\psi + R_\psi) - k_x k_y(I_\phi + R_\phi)]\right\}dk_x\,dk_y \; .$$

With (262) and (263) this may be written

(265) $$Y = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f^*}{j\omega\mu_o} \left[(k_1^2 - k_y^2) \left(2I_\psi - \frac{jf}{k_{z1}} \right) - k_x k_y \, 2I_\phi \right] dk_x \, dk_y .$$

Again Y will be determined in terms of the normalized constants

(266) $$\eta = \frac{k_x}{k_o}$$

(267) $$\xi = \frac{k_y}{k_o}$$

(268) $$\rho = C e^{-j\phi} = \frac{k_1}{k_o}$$

(269) $$A = k_o a$$

(270) $$B = k_o b$$

(271) $$D = k_o d \, .$$

Define also

(272) $$R = \sqrt{\rho^2 - \eta^2 - \xi^2}$$

(273) $$P = \sqrt{1 - \eta^2 - \xi^2}$$

$$F = 4\pi \sqrt{\frac{2B}{A}} \; \frac{\sin\left(\frac{\eta A}{2}\right)\cos\left(\frac{\xi B}{2}\right)}{\eta(\pi^2 - \xi^2 B^2)} \quad . \tag{274}$$

Then I_ψ and I_ϕ may be written

$$I_\psi = \frac{1}{k_o^2} I_\psi' \tag{275}$$

$$I_\phi = \frac{1}{k_o^2} I_\phi' \tag{276}$$

with I_ψ' and I_ϕ' given by

$$I_\psi' = \frac{\begin{vmatrix} C_{11} & D_1 \\ C_{12} & D_2 \end{vmatrix}}{\begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix}} \tag{277}$$

$$I_\phi' = \frac{\begin{vmatrix} D_1 & C_{12} \\ D_2 & C_{22} \end{vmatrix}}{\begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix}} \tag{278}$$

where

$$D_1 = -\eta\xi\, e^{+jRD} \left(\frac{R+P}{2R}\right) jF \tag{279}$$

(280) $$D_2 = \frac{1}{2} e^{+jRD} \left\{ (1-\xi^2) + (\rho^2 - \xi^2) \frac{P}{R} \right\} jF$$

(281) $$C_{11} = (\rho^2 - \eta^2) P \cos RD + j(1-\eta^2) R \sin RD$$

(282) $$C_{12} = -\eta\xi(P \cos RD + jR \sin RD)$$

(283) $$C_{22} = (\rho^2 - \xi^2) P \cos RD + j(1-\xi^2) R \sin RD .$$

The admittance Y, normalized to the free-space characteristic admittance y_o is then found to be

(284) $$Y_n = \frac{Y}{Y_o} = -j \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F \left\{ (\rho^2 - \xi^2) \left(2I_\psi' - j \frac{F}{R} \right) - 2\eta\xi I_\phi' \right\} d\eta \, d\xi.$$

Equation (284) has been evaluated on the OSU IBM 7094. The integration is actually done after a change of variables

(285) $$\eta = \beta \cos \alpha$$

(286) $$\xi = \beta \sin \alpha$$

so that

$$(287) \quad Y_n = -j \frac{1}{(2\pi)^2} \int_{\beta=0}^{\infty} \int_{\alpha=0}^{2\pi} F \left[(\rho^2 - \beta^2 \sin^2 \alpha)\left(2I_\psi' - j\,\frac{F}{R}\right) - 2\beta^2 \sin\alpha \cos\alpha \, I_\phi' \right] \beta \, d\beta \, d\alpha \, .$$

After this change, only one infinite integral must be evaluated, instead of two. Simpson's rule is used. The integral is done as an iterated integral, the integration on α being done first. The integration on β is done over a finite range, where the upper limit is chosen so the range of integration includes all values of β for which the integrand has a significant value. The method of choosing the limit is to test the integrand at successive increments of β, and if its value is small enough to contribute less than 0.0001 times the value of the integral for five increments in a row, the integration is terminated.

Numerical results were obtained for the case $A = \frac{\pi}{2}$, $B = \pi$, and $D = \pi$ and are shown in Figs. 31, 32, and 33. As expected, for k_1 with a large imaginary part, the admittance is the same as for an infinite lossy medium. For k_1 a real number, corresponding to a lossless dielectric, the admittance fluctuates rapidly as a function of k_1. Except near $k_1 = 0$, the admittance is seen to be quite similar to the admittance per unit length of the infinite slot.

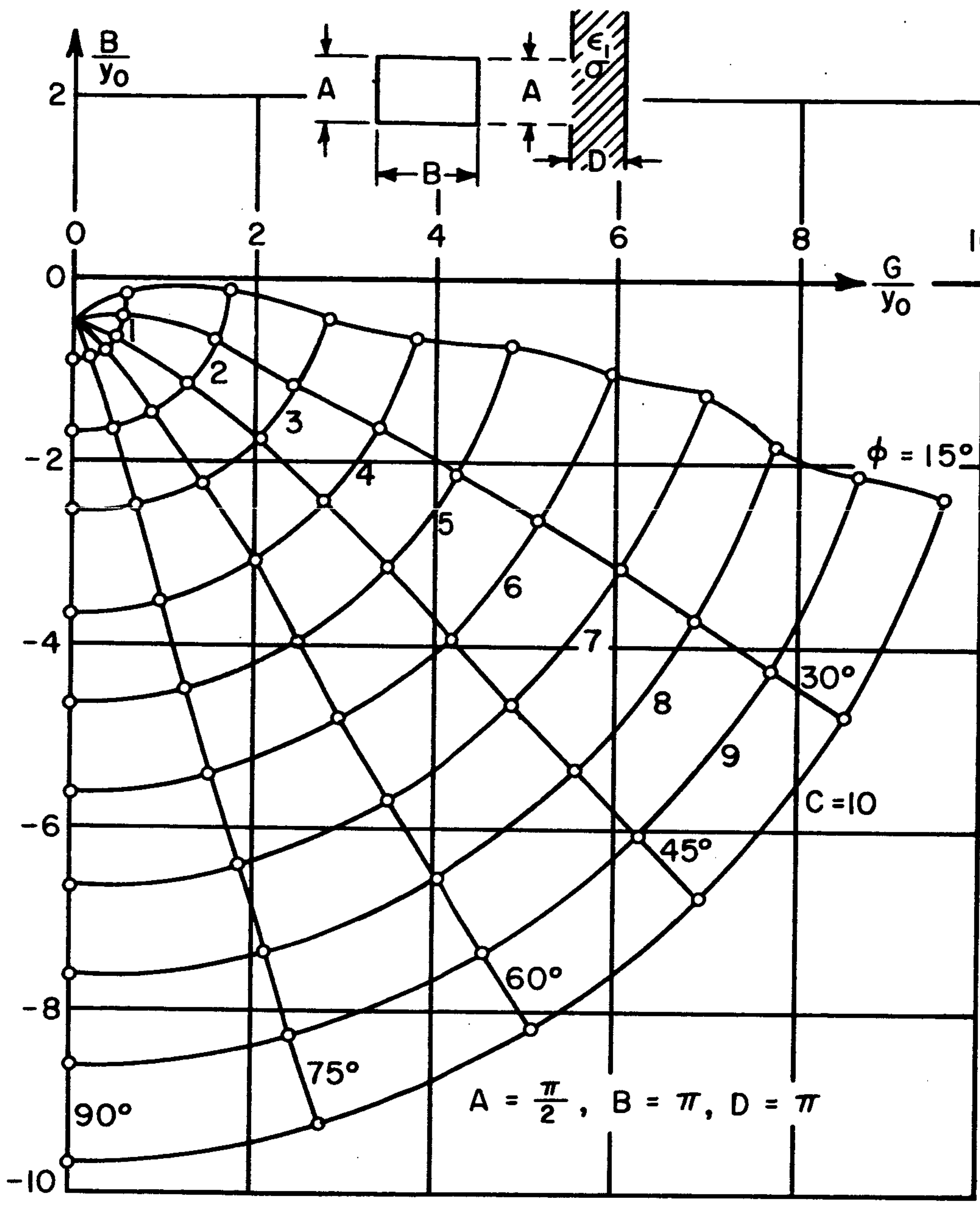

Fig. 31--The normalized admittance $\frac{Y}{Y_o}$ for a finite slab

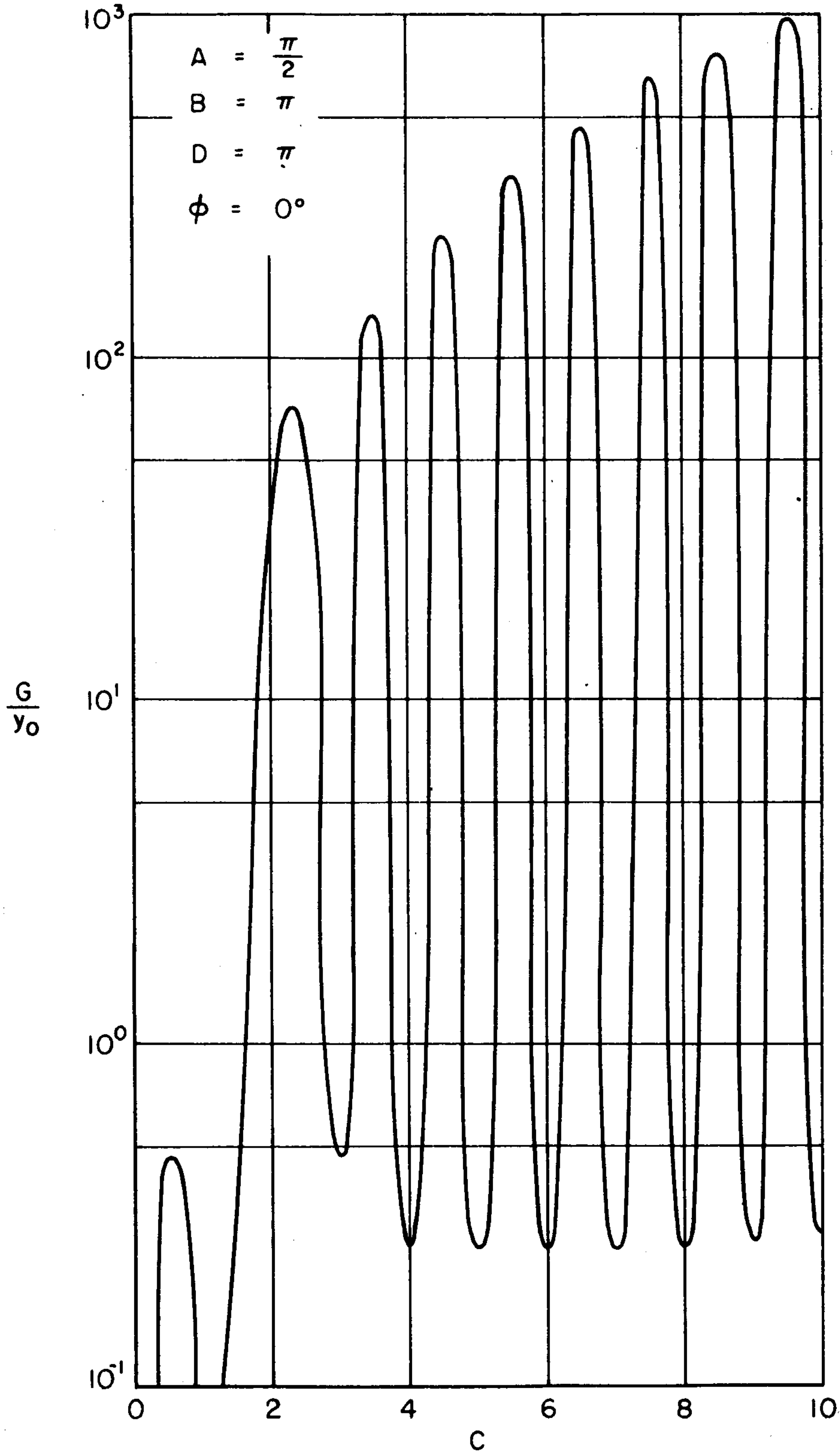

Fig. 32--The normalized admittance $\frac{Y}{Y_0}$ for a finite slab

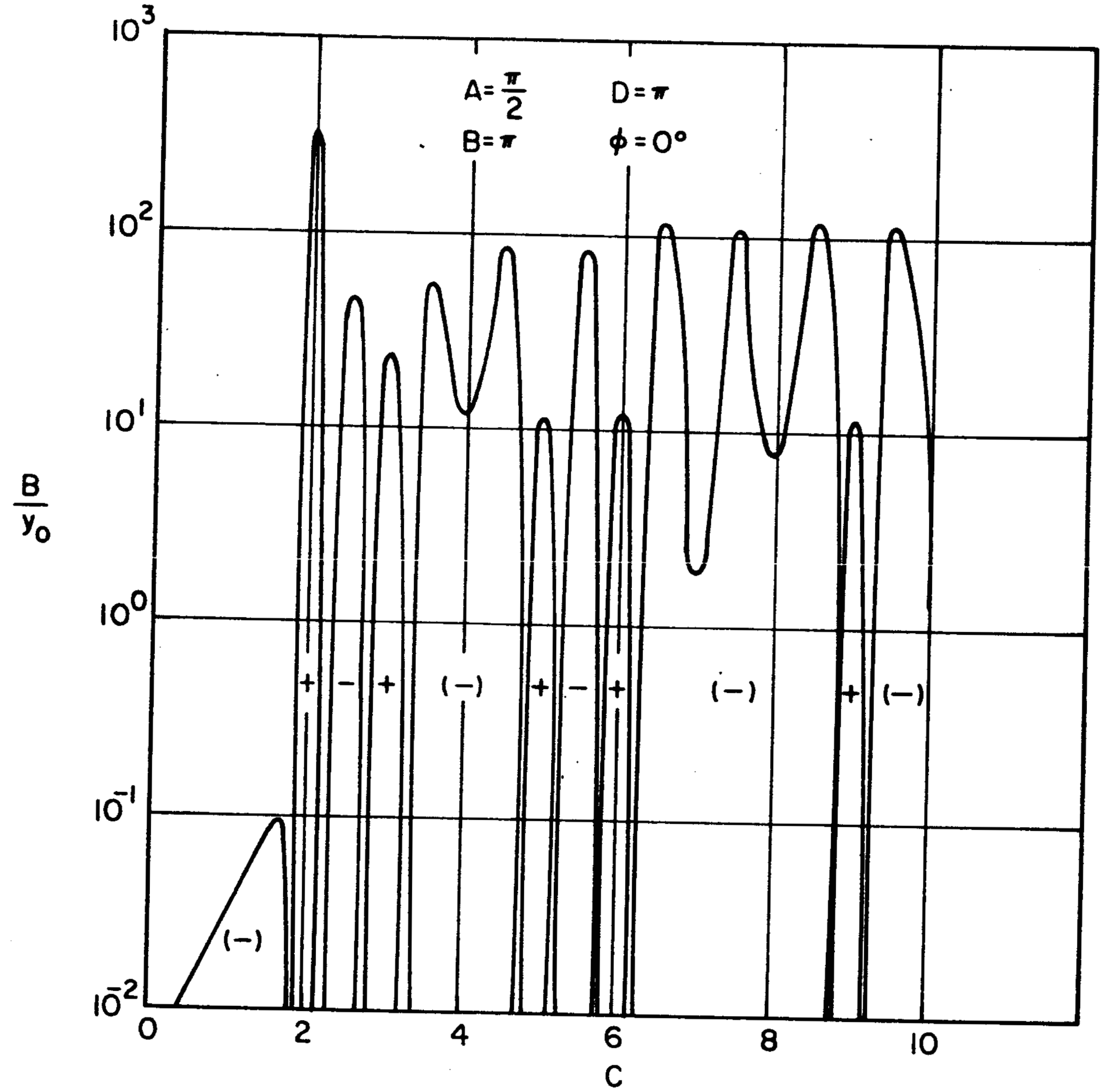

Fig. 33--The normalized admittance $\frac{Y}{Y_o}$ for a finite slab

CHAPTER V
CONCLUSIONS

The admittance has been found for an infinite slot and a rectangular aperture, as a function of the complex propagation constant of the lossy medium.

The admittance of an infinite slot radiating into a lossy half-space is shown in Fig. 3. Figure 3 is best interpreted as showing the admittance as a function of ϵ_1 and σ_1, for fixed aperture size and fixed frequency. The dependence of Y on aperture size or on frequency can be calculated from Fig. 3. It is noted that the behavior of Y near $\omega = 0$ is markedly different for $\sigma_1 = 0$ than for $\sigma_1 \neq 0$.

The admittance of an infinite slot radiating through a lossy slab into free-space has been found and the results are given in Figs. 6 through 23. As would be expected, when the slab becomes sufficiently lossy, the admittance is identical with that for an infinite lossy medium. For a low-loss slab, the admittance fluctuates rapidly as a function of the propagation constant k_1. For lossless k_1, the admittance undergoes resonances which are more pronounced for the higher values of k_1. This behavior is to be expected because, viewing the slab as a section of transmission line between the aperture and the free-space region, it is seen that the higher the value of k_1 the worse is the impedance mismatch at the interface.

The admittance of a rectangular aperture antenna has been found in Chapter IV. The results for an infinite lossy medium are shown in Figs. 25 through 29, for three sizes of aperture. The admittance for the rectangular aperture is seen to be similar to that of the infinite slot. One difference, however, is seen near C = 0, where the admittance of the infinite slot goes to zero but that of the rectangular aperture has a finite negative susceptance.

The admittance of a rectangular aperture radiating through a lossy slab has been found. Numerical results are plotted in Figs. 31, 32, and 33, for the case $A = \frac{\pi}{2}$, $B = \pi$, $D = \pi$. Again the admittance is identical with that of an infinite lossy medium when k_1 has a large complex value. As with the infinite slot antenna, the admittance for real k_1 undergoes resonances which grow in amplitude as k_1 becomes larger.

It is noted that in all cases the aperture admittance is inductive when the medium is highly lossy. This is to be expected since for a lossy medium the fields do not penetrate the medium to any apprecialbe depth. The aperture fields are then essentially the same as they would be if the half-space were replaced with a continuation of the waveguide, filled with the lossy medium. The terminating admittance would then be simply the characteristic admittance of the filled guide, which is inductive.

For values of k_1 corresponding to a medium with large loss, the admittance of the infinite slot was found to be given by

$$\frac{A}{Y_o} Y = \frac{k_1}{k_o} + j \frac{2}{\pi}$$

and the admittance of the rectangular aperture by

$$\frac{Y}{Y_o} = \frac{k_1}{k_o} .$$

The admittance of the slot or the rectangular aperture radiating through a slab was found to be double-valued, as a function of C, when the slab is lossless. This behavior is to be expected, since the slab acts as a transmission line joining the aperture to the free-space region. As C changes, the electrical length of the line changes, and the input admittance at the aperture varies according. As would be true of an ordinary transmission line, the admittance is double valued, (it loops on a Smith Chart).

21 March 1967

ERRATA FOR REPORT 1691-5, R. T. Compton, Jr., "The Admittance of Aperture Antennas Radiating Into Lossy Media," 15 March 1964, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Grant Number NsG-448 for National Aeronautics and Space Administration.

The numerical results as given in Report 1691-5 for the aperture admittances of the infinite slot and the rectangular aperture are incorrect for the cases of lossless or low loss slabs covering the apertures. Specific errors in Report 1691-5 include:

1) The results given in the following figures for the infinite slot covered by a lossless slab ($\phi = 0$) are incorrect — Figures 7, 9, 11, 13, 15, 17, 19, 21, 23.

2) The infinite slot results for the $\phi = 15°$ slab as plotted in the following figures may be incorrect. Also the accuracy is in doubt for some of the curves for $\phi = 30°, 45°$ in the following figures. Figures 8, 10, 12, 14, 16, 18, 20, 22.

3) The rectangular aperture results for the 15° slabs as plotted in Fig. 31 may be inaccurate.

4) The results given in Figs. 32 and 33 for the rectangular aperture covered by a lossless slab ($\phi = 0$) are incorrect.

5) The observations and conclusions by the author on pages 46, 47, 88, 92, 93 and 94 of Report 1691-5 regarding the oscillatory and double valued behavior of the results for low loss slabs should be disregarded.

The basic analysis for the infinite slot and the rectangular aperture are correct as given in 1691-5. The details of the derivation for the rectangular aperture have been checked and all equations including the final equation (287) are correct. The source of error in obtaining numerical results from the analyses is the neglect of surface wave which occur in the integrand of the final equations.

Correct numerical results have been obtained for lossless slabs by developing a computer program which locates the surface wave poles, numerically integrates between the poles, and evaluates the contribution of the surface wave pole residues to the aperture admittance. The surface wave pole analysis, the highlights of the associated computer program and the corrected theoretical results for the aperture admittance of the rectangular waveguide are given in Report 1691-21.